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Homological algebra

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Abstract

This thesis gives some results in the topics of modules and categories as they directly relate with the functor Ext used in homological algebra.

Acknowledgements

To Dr. Gentle and Jessica for all the help you gave to make this possible.

Contents

Chapter 1

Introduction:

Homology is a generic process that is used to give the algebraic structure of sequences of modules or abelian groups to mathematical objects such as topological spaces. The approach this thesis takes is through homological algebra, which has direct applications to algebraic geometry, representation theory, mathematical physics, and partial differential equations. Homological algebra relies heavily on a concept known as category theory, affectionately referred to as abstract nonsense due to the fact that many are lost from the level of abstraction.

Homological algebra can trace its origins back to the end of the 19th century. Henri Poincare and David Hilbert are consider to be the chief investigators in the research of homology. The development of homological algebra is closely linked to the development of category theory which is presented in this thesis.

Samuel Eilenberg and Sanders MacLane introduced categories, functors, and natural transformations in their development of algebraic topology. This gave rise to an axiomatic approach to homology from intuitive and geometric ones. The development of category theory was first driven by the computational needs of the homological algebra.

This thesis does not go into the full theory of homological algebra. It begins by introducing the basic ideas needed to create what are known as derived functors, the central idea in homological algebra. We will then introduce the notion of module theory as it deals directly with the theory of homology. Hom sets are the first major idea brought up in that chapter. Followed by sequences and projective and injective modules.

The chapter that discusses categories is narrowly focused on the ideas needed to develop a specific functor known as Ext. It should not be seen as even an overview of the subject of categories. The theory begins by defining things in a more general sense to give the reader a feel for the approach that category theory takes. However, this view of categories is unneeded to discuss the complex ideas presented. Thus definitions are quickly changed to be specific to certain categories to give the reader a more familiar setting to work with.

From here the functor Ext is developed in three different approaches. One in the setting of Sets and module theory. The second approach is uses the idea of the projective resolution of a sequence. Finally, Ext is created through injective modules. All three approaches are then shown to be equivalent and shed light on the structure of Ext.

Finally, we sum up this entire process through directed graphs. First presented in Chapter 3, directed graphs are used to illustrate concepts in categories. Later in chapter 5, directed graphs are used to give the reader another example on how to derive the Ext functor.

Chapter 2

Modules:

2.1 Definition

For this thesis the reader will need to be familiar with several concepts from the theory of modules and categories. We will begin with modules.

Definition 2.1 Let R be a ring. A *left R-module* or a *left module over R* is a set M together with:

(1) a binary operation + on M under which M is an abelian group, and

(2) an action of R on M (that is, a map $R \times M \to M$) denoted by rm, for all $r \in R$ and for all $m \in M$ which satisfies:

(i) $(r+s)m = rm + sm$, for all $r, s \in R$, $m \in M$

(ii) $(rs)m = r(sm)$ for all $r, s \in R$, $m \in M$

(iii) $r(m+n) = rm + rn$ for all $r \in R$, $m, n \in M$

(iv) If the ring R has a 1 then, $1m = m$ for all $m \in M$

$\diamondsuit$

Modules behave much like vector spaces. Unless otherwise stated assume that all modules are left modules and rings have unity. Most of the sequences dealt with throughout this thesis will be over modules.

Example 2.2 The integers acting on $\mathbb{Z}_p$ will be a left $\mathbb{Z}$-module. In fact, the integers acting on any abelian group will form a left $\mathbb{Z}$-module by defining nx as repeated addition, so $nx = x + x + \cdots + x$ where $n \in \mathbb{N}$. For $-n$ define $-nx = -(nx)$. □

Definition 2.3 Given that M and N are modules, a map $f : M \to N$ is a module homomorphism if for any $s, r \in R$ and $m, n \in M$, $f(rm + sn) = rf(m) + sf(n)$. ♢

Note: If R has unity, then f is a group homomorphism.

2.2 Hom Sets

The next major idea that will be that of *Hom* Sets. Given A and B both modules over R, $Hom_R(A, B)$ denotes the set of all module homomorphisms from A to B. Given two homomorphisms from A to B, ($\phi : A \to B$ and $\psi : A \to B$) we can define addition on the set $Hom_R(A, B)$ by $\phi + \psi$ as the morphism given by $(\phi + \psi)(a) = \phi(a) + \psi(a)$. Next we would like to check to see if it is a module homomorphism. So given $a \in A, b \in B$, and $r \in R$

$$
\begin{aligned}
(\phi + \psi)(a + rb) &= \phi(a + rb) + \psi(a + rb) && \text{definition of } \phi + \psi \\
&= \phi(a) + r\phi(b) + \psi(a) + r\psi(b) && \phi, \psi \text{ are homomorphisms} \\
&= \phi(a) + \psi(a) + r(\phi(b) + \psi(b)) && \\
&= (\phi + \psi)(a) + r(\phi + \psi)(b) && \text{definition of } \phi + \psi
\end{aligned}
$$

It follows that since $+_R$ is abelian $+$ for$Hom_R(A,B)$ is abelian as well and $Hom_R(A,B)$ forms an abelian group. In the next sections of this text we will suppress the R on the $Hom_R(-,-)$ notation when it is not necessary.

Theorem 2.4 $Hom_{\mathbb{Z}}(\mathbb{Z}_n, A) \cong A_{(n)}$ where A is an abelian group, and $A_{(n)} = \{a \in A | na = 0\}$.

Proof: We can define a mapping $\Gamma : Hom_{\mathbb{Z}}(\mathbb{Z}_n, A) \longrightarrow A_{(n)}$ by $\Gamma(\phi) = \phi(1)$. Consider

$$\begin{aligned} n\phi(1) &= \phi(1) + \phi(1) + \cdots + \phi(1) && \text{(n times)} \\ &= \phi(1 + 1 + \cdots + 1) && \text{(since } \phi \text{ is a homomorphism)} \\ &= \phi(n) \\ &= \phi(0) \end{aligned}$$

Since ϕ is a group homomorphism it preserves the additive identity, so $n\phi(1) = 0$. Resulting in $\phi(1) \in A_{(n)}$. Now consider

$$\begin{aligned} \Gamma(\phi + \psi) &= (\phi + \psi)(1) \\ &= \phi(1) + \psi(1) \\ &= \Gamma(\phi) + \Gamma(\psi) \end{aligned}$$

Thus Γ a homomorphism of abelian groups.

Define a map $\Lambda : A_{(n)} \longrightarrow Hom_{\mathbb{Z}}(\mathbb{Z}_n, A)$ as follows. For each $a \in A_{(n)}$, $\Lambda(a) = \phi_a$ where $\phi_a(k) = ka$. Clearly, ϕ_a is a homomorphism $\phi_a : \mathbb{Z}_n \longrightarrow A$.

Given $\phi \in Hom_{\mathbb{Z}}(\mathbb{Z}_n, A)$

$$\begin{aligned} \Lambda\Gamma(\phi) &= \Delta(\phi(1)) \\ &= \phi_{\phi(1)} \end{aligned}$$

Now for each $k \in A$

$$\begin{aligned}\phi_{\phi(1)}(k) &= k\phi(1) \\ &= \phi(1) + \phi(1) + \cdots + \phi(1) \\ &= \phi(1 + 1 + \cdots + 1) \\ &= \phi(k)\end{aligned}$$

Thus $\Lambda\Gamma(\phi) = \phi$. Now let $a \in A_{(n)}$

$$\begin{aligned}\Gamma\Lambda(a) &= \Lambda(\phi_a) \\ &= \phi_a(1) \\ &= a1 \\ &= a\end{aligned}$$

It then follows that Γ is an isomorphism $Hom_{\mathbb{Z}}(\mathbb{Z}_n, A) \cong A_{(n)}$. Since $\Lambda\Gamma$ is the identity on $Hom_{\mathbb{Z}}(\mathbb{Z}_n, A)$ and $\Gamma\Lambda$ is the identity on $A_{(n)}$. ∎

Example 2.5 $Hom_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_m)$

Here $(\mathbb{Z}_m)_n = \{k \in \mathbb{Z}_m | nk = 0\}$, and

$$nk = 0 \mod m \tag{2.1}$$

$$\Leftrightarrow \quad m \text{ divides } nk \tag{2.2}$$

$$\Leftrightarrow \quad \tfrac{m}{d} \text{ divides } \tfrac{n}{d}k \qquad \text{for } d = gcd(n, m) \tag{2.3}$$

$$\Leftrightarrow \quad \tfrac{m}{d} \text{ divides } k \qquad \text{since } gcd\left(\frac{m}{d}, \frac{n}{d}\right) = 1 \tag{2.4}$$

Resulting in $(\mathbb{Z}_m)_n = \{l\frac{m}{d} | l \in \mathbb{Z}\} = \left\langle \frac{m}{d} \right\rangle$ which is a cyclic subgroup of $\mathbb{Z}_m$ of order d generated by $\frac{m}{d}$. Hence $\left\langle \frac{m}{d} \right\rangle \cong \mathbb{Z}_d$, since any cyclic group of order d is isomorphic to $\mathbb{Z}_d$. Thus by Theorem 2.4, a generic $\phi \in Hom_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_m)$ will have the form $\phi(k) = l\left(\frac{m}{d}\right)$ for a fixed l where $0 \leq l < d$ and $Hom_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_d$. □

Example 2.6 For $B_1 \oplus B_2$, the direct sum of modules B_1 and B_2, and two module homomorphisms $\phi_i \in Hom(A, B_i)$ for $i = 1, 2$ we can define a $\phi \in Hom(A, B_1 \oplus B_2)$ by

$$\phi(a) = (\phi_1 a, \phi_2 a) = \begin{bmatrix} \phi_1(a) \\ \phi_2(a) \end{bmatrix}$$

Denote ϕ by $\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$. Similarly, for $\psi_i \in Hom(A_i, B)$ for $i = 1, 2$ we can define $\psi \in Hom(A_1 \oplus A_2, B)$ by

$$\psi(a_1, a_2) = \psi_1(a_1) + \psi_2(a_2).$$

ψ will be denoted by $[\psi_1, \ \psi_2]$. More generally, if $\phi_i \in Hom(A, B_i)$ for $i = 1, ..., n$, define

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix} \text{ by } \phi(a) = \begin{bmatrix} \phi_1(a) \\ \phi_2(a) \\ \vdots \\ \phi_n(a) \end{bmatrix}$$

and if $\psi_i \in Hom(A_i, B)$ for $i = 1, ..., n$, define $\psi = [\psi_1, ..., \psi_n]$ by

$$[\psi_1, ..., \psi_n] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \sum_{i=1}^{n} \psi_i(a_i).$$

This will imply that, for finite sums, $Hom(\oplus_i A_i, B) \cong \oplus_i Hom(A_i, B)$. As well as $Hom(A, \oplus_j B_j) \cong \oplus_j Hom(A, B_j)$.

Theorem 2.7 For $i, j \in \mathbb{N}$, $Hom(\oplus_i A_i, B) \cong \oplus_i Hom(A_i, B)$ and $Hom(A, \oplus_j B_j) \cong \oplus_j Hom(A, B_j)$.

Proof: For $\phi_j \in Hom(A, B_j)$ with $j = 1, ..., n$, define $\phi \in Hom(A, \oplus_j B_j)$, (where elements of $\oplus_j B_j$ are written as column vectors) by

$$\phi(a) = \begin{bmatrix} \phi_1(a) \\ \phi_2(a) \\ \vdots \\ \phi_n(a) \end{bmatrix}$$

Define a mapping $\pi_k : \oplus_j B_j \longrightarrow B_k$, called the canonical (natural) projection, by

$$\pi_k \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = b_k,$$

where b_k is the kth component of the vector $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$. Then

$\oplus_j Hom(A, B_j) \cong Hom(A, \oplus_j B_j)$ by $\begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix} \mapsto \phi$ defined as above, with the inverse $\phi \mapsto \begin{bmatrix} \pi_1 \circ \phi \\ \vdots \\ \pi_n \circ \phi \end{bmatrix}$, which are both clearly group homomorphisms. With this isomorphism, $\phi \in Hom(A, \oplus_j B_j)$ is usually denoted as the column vector $\begin{bmatrix} \phi_1 \\ \vdots \\ \phi_n \end{bmatrix}$ and regarded as an element of $\oplus_j Hom(A, B_j)$.

Now for $\psi_i \in Hom(A_i, B)$, with $i = 1, ..., m$, define $\psi \in Hom(\oplus_i A_i, B)$, (where elements of $\oplus_i A_i$ are given as column vectors) by

$$\psi \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} = \sum_{i=m}^{m} \psi_i(a_i).$$

We can then define a mapping, $i_k : A_k \longrightarrow \oplus_i A_i$, called the the canonical (natural) injection by

$$i_k(a) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Where a occurs in the kth component of the vector. Then
$\oplus_i Hom(A_i, B) \cong Hom(\oplus_i A_i, B)$ by $(\psi_1, ..., \psi_m) \mapsto \psi$, defined as above, and with the inverse $\psi \mapsto (\psi \circ i_1, ..., \psi \circ i_m)$ (which are again both group homomorphisms). With this isomorphism, the elements $\psi \in Hom(\oplus_i A_i, B)$ are denoted as row vectors and regarded as elements of $\oplus_i Hom(A_i, B)$. ∎

Now we will discuss some basic ideas for elements in $Hom_R(A, B)$. Given $f \in Hom(A, B)$, we define the image of f, denoted as $\operatorname{Im} f$, by
$\operatorname{Im} f = \{f(a) \in B | a \in A\}$. The kernel of f, denoted as $\ker f$, is all of the elements in A that are sent to the identity in B by the homomorphism f, (ie $\ker f = \{a | f(a) = 0\}$). Finally, the cokernel of f (the dual notion to kernel), denoted as $\operatorname{cok} f$, is defined as the module B modded out by $\operatorname{Im} f$ (ie. $\operatorname{cok} f = B/\operatorname{Im} f$).

In the above theorem, we saw the canonical injection map for external direct sums. Another kind of natural injection map could involve submodules. For a given module, M, with submodule, N, the injection map, $i : N \longrightarrow M$, is $i(x) = x$. In fact, the injection map is any mapping that satisfies the following universal property.

Property 2.8 For $A \xrightarrow{\alpha} B$, the inclusion $\ker\alpha \xrightarrow{i} A$ has the following universal property:

(i) $0 = \alpha i$

(ii) if $0 = \alpha\gamma$ with $\gamma : M \longrightarrow A$ then there exists a unique $\lambda : M \longrightarrow \ker\alpha$ such that $\gamma = i\lambda$. Diagrammatically, this is

$$\begin{array}{ccccc} \ker\alpha & \xrightarrow{i} & A & \xrightarrow{\alpha} & B \\ \uparrow{\scriptstyle \exists!\lambda} & \nearrow{\scriptstyle \gamma} & & & \\ M & & & & \end{array}$$

Proof:

i) Clear, since for $a \in \ker\alpha$, $\alpha i(a) = \alpha(a) = 0$

ii) Suppose $\alpha\gamma = 0$, so $\alpha\gamma(m) = 0$ for all $m \in M$. Thus $\gamma(m) \in \ker\alpha$. Now let $\lambda : M \longrightarrow \ker\alpha$ be $\lambda(m) = \gamma(m)$. Then

$$\begin{aligned} i\lambda(m) &= \lambda(m) \\ &= \gamma(m) \end{aligned}$$

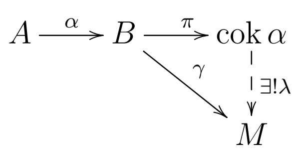
For uniqueness, suppose $i\lambda = \gamma$, then for any $m \in M$, $i\lambda(m) = \gamma(m)$, so $\lambda(m) = \gamma(m)$ since i is just the inclusion map. ∎

Similarly, the earlier definition of the projection map was given for internal direct sum. However, given a module, M, with submodule, N, the projection map, $\pi : M \longrightarrow M/N$, is given by $\pi(x) = \overline{x} = x + M$. In fact, the canonical projection map is any mapping that satisfies the following universal property.

Property 2.9 For $A \xrightarrow{\alpha} B$, the projection $B \xrightarrow{\pi} \operatorname{cok}\alpha$ has the following universal property:

(i) $0 = \pi\alpha$

(ii) if $0 = \gamma\alpha$, $\gamma : B \longrightarrow M$, then there exists a unique $\lambda : \operatorname{cok}\alpha \longrightarrow M$ such that $\gamma = \lambda\pi$. Diagrammatically, this is



Proof:

i) Clear, since $\pi\alpha(x) = \overline{\alpha(x)} = \overline{0}$ because $\alpha(x) \in \operatorname{Im}\alpha$.

ii) The elements of $\operatorname{cok}\alpha$ are $\pi(b)$ with $b \in B$, so we must define λ so that $\lambda(\pi(b)) = \gamma(b)$. Such a λ would then be unique. To establish λ is well-defined, suppose $\pi(b_1) = \pi(b_2)$. Then $\pi(b_1 - b_2) = 0$, implying that $(b_1 - b_2) \in \ker\pi$. However, $\ker\pi = \operatorname{Im}\alpha$ and thus $b_1 - b_2 = \alpha(a)$ for some $a \in A$.

$$\begin{aligned} 0 &= \gamma\alpha(a) \\ &= \gamma(b_1 - b_2) \\ &= \gamma(b_1) - \gamma(b_2) \\ \gamma(b_1) &= \gamma(b_2) \end{aligned}$$

This results in $\lambda(\pi(b_1)) = \gamma(b_1) = \gamma(b_2) = \lambda(\pi(b_2))$. Meaning that there exists a module homomorphism that is uniquely determined by γ. ∎

We finally then define some notation that will be used throughout the thesis. Given a morphism $A \xrightarrow{\varepsilon} B$ we can define a new morphism as follows: For each module X, $\varepsilon^* : Hom(X, A) \longrightarrow Hom(X, B)$ is defined as $\varepsilon^*(\phi) = \varepsilon\phi$ for each $\phi \in Hom(X, A)$. Dually, $\varepsilon_* : Hom(B, X) \longrightarrow Hom(A, X)$ is defined as $\varepsilon_*(\psi) = \psi\varepsilon$.

2.3 Sequences

Using the canonical injection and projection maps for an associated homomorphism f, denoted as i_f and π_f respectively, we can create the following sequence.

$$\ker f \xrightarrow{i_f} A \xrightarrow{f} B \xrightarrow{\pi_f} \operatorname{cok} f$$

By definition we have $\ker f = \operatorname{Im} i_f$ and $\ker \pi_f = \operatorname{Im} f$, also i_f is a monomorphism and π_f is an epimorphism. Such a sequence is said to be exact at A and B.

Definition 2.10 The pair of homomorphism $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is said to be *exact* at Y if $\operatorname{Im} \alpha = \ker \beta$. Furthermore, a sequence $\cdots \longrightarrow X_{n-1} \xrightarrow{\phi_n} X_n \xrightarrow{\phi_{n+1}} X_{n+1} \longrightarrow \cdots$ of homomorphism is said to be an *exact sequence* if it is exact at every X_n between a pair of homomorphisms. $\diamondsuit$

Note: When exact at X_n for $a \in X_n, \phi_{n+1}(a) = 0$ if and only if $a = \phi_n(b)$ for some $b \in X_{n-1}$, implying that $\phi_{n+1} \circ \phi_n = 0$.

Theorem 2.11 Let A,B,C be R-modules, then:

(i) the sequence $0 \longrightarrow A \xrightarrow{\psi} B$ is exact if and only if ψ is injective

(ii) the sequence $B \longrightarrow C \xrightarrow{\phi} 0$ is exact if and only if ϕ is surjective where 0 is the trivial module ($0 = \{0\}$).

Proof: The uniquely defined map $0 \longrightarrow A$ has image 0 in A. This will be $\ker \psi$ if and only if ψ is injective. Similarly, the kernel of the uniquely defined zero homomorphism $C \longrightarrow 0$ is all of C, which is the image ϕ if and only if ϕ is surjective. ∎

Corollary 2.12 The sequence $0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0$ is exact if and only if ψ is injective, ϕ is surjective, and $\mathrm{Im}\ \psi = \ker \phi$.

Definition 2.13 The sequence of the previous corollary, $0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0$, is called a *short exact sequence*.

Theorem 2.14 For $K \xrightarrow{\nu} A \xrightarrow{\alpha} B$, ν induces an isomorphism $K \longrightarrow \ker \alpha$ if and only if ν satisfies the following universal property:

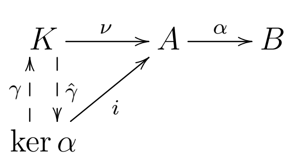
(i) $\alpha\nu = 0$

(ii) if $\alpha\gamma = 0$, then there exists a unique λ such that $\gamma = \nu\lambda$

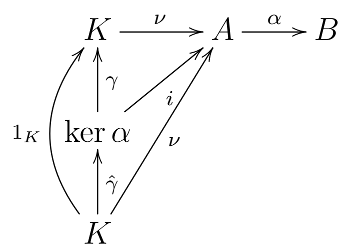
Proof:

Assuming ν induces an isomorphism, $K \xrightarrow{\sim} \ker \alpha$, then we get i) and ii) from the universal property of the inclusion (Property 2.8).

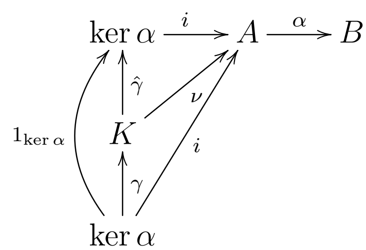
Assuming ν satisfies this universal property, then there is a unique γ with $i = \nu\gamma$ and using the universal property of i we get another unique $\hat{\gamma}$ such that $\nu = i\hat{\gamma}$ giving the following diagram.



Using γ to be ν the unique λ would have to be the identity on K.



Which gives $\nu\gamma\hat{\gamma} = \nu$, hence $\gamma\hat{\gamma} = 1_K$. Now using γ to be i we get λ to be the identity on $\ker\alpha$.



Giving $i\hat{\gamma}\gamma = i$ and $\hat{\gamma}\gamma = 1_{\ker\alpha}$. Implying that $\hat{\gamma} = \gamma^{-1}$, so γ is an isomorphism.

▮

Theorem 2.15 For $A \xrightarrow{\alpha} B \xrightarrow{\nu} L$, ν induces an isomorphism $L \longrightarrow \operatorname{cok}\alpha$ if and only if ν satisfies the following universal property:

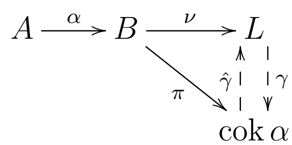
(i) $\nu\alpha = 0$

(ii) if $\gamma\alpha = 0$, then there exists a unique λ such that $\gamma = \lambda\nu$

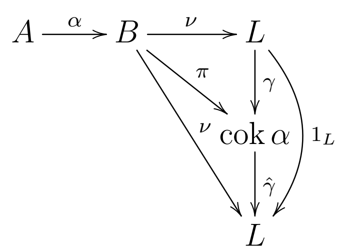
Proof:

Assuming that ν induces an isomorphism, $L \xrightarrow{\sim} \operatorname{cok}\alpha$, we then get i) and ii) as consequences of the the universal property of the projection (Property 2.9).

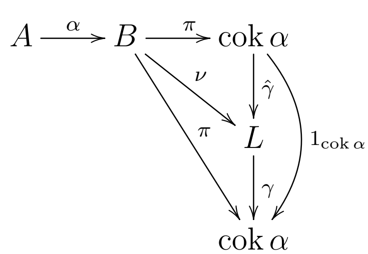
Now assuming that ν satisfies this universal property, there exists a unique γ with $\pi = \gamma\nu$ and using the universal property of π we get another unique $\hat{\gamma}$ such that $\nu = \hat{\gamma}\pi$. Giving the following diagram.



Thus using γ to be ν the unique λ would have to be the identity on L.



Which gives $\hat{\gamma}\gamma\nu = \nu$, hence $\hat{\gamma}\gamma = 1_L$. Now using γ to be π, we get λ to be the identity on $\operatorname{cok}\alpha$.



Giving $\gamma\hat{\gamma}\pi = \pi$ and $\gamma\hat{\gamma} = 1_{\operatorname{cok}\alpha}$. Implying that $\hat{\gamma} = \gamma^{-1}$, so γ is an isomorphism.

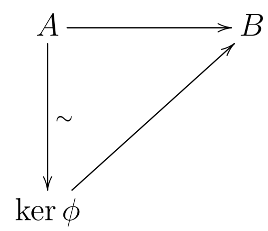
∎

While these was presented as a property of the kernel and cokernel of R-module homomorphisms, nowhere did we use the definition of kernel, cokernel, projection and inclusion morphism directly; only the universal property of i and π was needed. Meaning we can actually use this universal property to define the kernel or cokernel of a general morphism. This will be discussed more in the next chapter. Furthermore, this definition would imply that the kernel is unique if it exists.

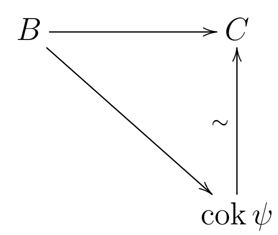
Returning to a short exact sequence,

$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0$$

we get, ψ induces the isomorphism



and ϕ induces the isomorphism



Giving us the following theorem.

Theorem 2.16 Let

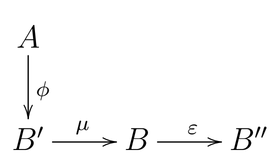
$$0 \longrightarrow B' \xrightarrow{\mu} B \xrightarrow{\varepsilon} B''$$

be an exact sequence of R-modules. For every R-module A the induced sequence

$$0 \longrightarrow Hom(A, B') \xrightarrow{\mu*} Hom(A, B) \xrightarrow{\varepsilon*} Hom(A, B'')$$

is exact. Where μ^* is defined by $\mu^*(\phi) = \mu\phi$.

Proof: Assume that $\mu\phi$ in the diagram



is the zero map. Then $\mu\phi(a) = 0$ for all a, which implies that $\phi(a) = 0$ for all a, since μ is injective. Giving $\phi : A \longrightarrow B'$ as the zero map, so μ^* is injective.

A map in $\operatorname{Im}\mu^*$ is of the form $\mu\phi$. Since $\varepsilon\mu = 0$, then $\varepsilon\mu\phi$ is the zero map, thus $\ker\varepsilon^* \supset \operatorname{Im}\mu^*$. Finally, we show that $\ker\varepsilon^* \subset \operatorname{Im}\mu^*$, consider the diagram

$$\begin{array}{ccccc} & & A & & \\ & \swarrow^{\lambda} & \downarrow{\scriptstyle\psi} & & \\ B' & \xrightarrow{\mu} & B & \xrightarrow{\varepsilon} & B'' \end{array}$$

Suppose $\psi \in \ker\varepsilon^*$, thus $\varepsilon\psi$ is the zero map. Then, by Theorem 2.14, we get there exists $\lambda : A \longrightarrow B'$ such that $\mu\lambda = \psi$. Giving $\psi = \mu^*(\lambda)$ and thus $\operatorname{Im}\mu^* = \ker\varepsilon^*$. ∎

Theorem 2.17 Let

$$A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \longrightarrow 0$$

be an exact sequence of R-modules. For every R-module B the induced sequence

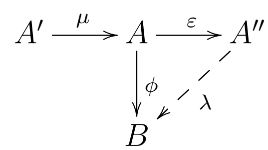
$$0 \longrightarrow Hom(A'', B) \xrightarrow{\varepsilon*} Hom(A, B) \xrightarrow{\mu*} Hom(A', B)$$

is exact. Where ε_* is defined as $\varepsilon_*(\phi) = \phi\varepsilon$.

Proof: Suppose $\phi \in \ker\varepsilon_*$, then $\varepsilon_*(\phi) = 0$. Since ε is onto, for any $a'' \in A''$ we get $\varepsilon(a) = a''$ for some $a \in A$. Which implies that $\phi(a'') = \phi(\varepsilon(a)) = \phi\varepsilon(a) = 0$ and thus $\phi = 0$. So $\ker\varepsilon_* = \{0\}$ and ε_* is injective. For

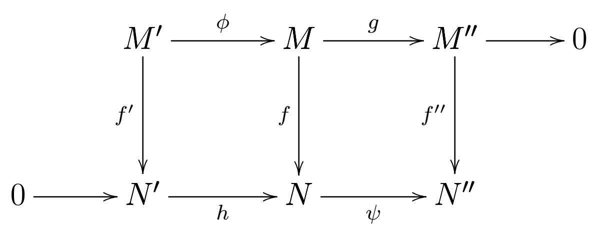
$$\begin{aligned} \mu_*\varepsilon_*(\phi) &= \mu_*(\phi\varepsilon) \\ &= \phi\varepsilon\mu \\ &= \phi 0 \\ &= 0 \end{aligned}$$

Thus $\operatorname{Im}\varepsilon_* \subseteq \ker\mu_*$. Considering the following diagram.

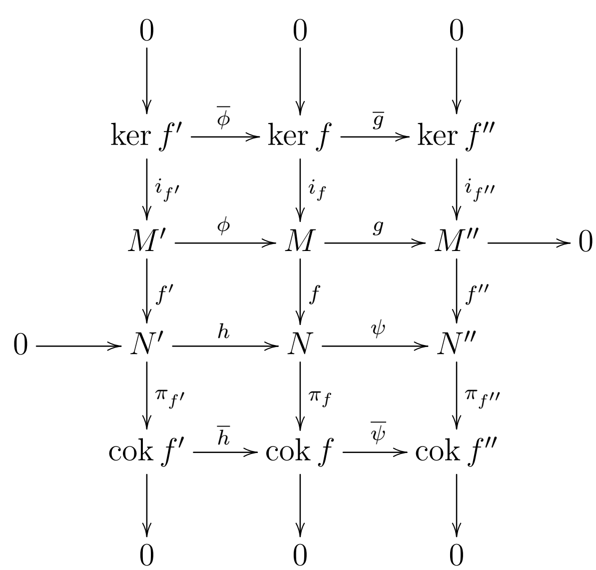


Suppose $\phi \in \ker\mu_*$, thus $\phi\mu$ is the zero. Then, by Theorem 2.15, there exists a $\lambda : A'' \longrightarrow B$ such that $\phi = \lambda\varepsilon$, and thus $\ker\mu_* = \operatorname{Im}\varepsilon_*$. ∎

Theorem 2.18 ***Snake Lemma:*** Given the following commutative diagram of R-modules with exact rows



there exists a map δ, called the connecting homomorphism, in the induced diagram



Where for f, f', f'', $\ker f$ denotes the kernel of f and $\operatorname{cok} f$ denotes the cokernel of f and the i are canonical inclusion module homomorphisms and the π are the canonical projection module homomorphisms and $\delta : \ker f'' \to \operatorname{cok} f'$ such that the sequence

$$0 \longrightarrow \ker f' \xrightarrow{\overline{\phi}} \ker f \xrightarrow{\overline{g}} \ker f'' \xrightarrow{\delta} \operatorname{cok} f' \xrightarrow{\overline{h}} \operatorname{cok} f \xrightarrow{\overline{\psi}} \operatorname{cok} f'' \longrightarrow 0$$

is exact and all sections of the diagram commute.

Proof: Since each i is the inclusion map it will thus be injective and each π is the projection map and thus surjective. Also, by definition of ker and cok, we have $\ker \pi_f = \operatorname{Im} f$ and $\ker f = \operatorname{Im} i_f$. The same argument can be made for the columns involving f' and f''. Thus all the columns

$$0 \longrightarrow \ker f \xrightarrow{i_f} M \xrightarrow{f} N \xrightarrow{\pi_f} \operatorname{cok} f \longrightarrow 0$$

are exact for f, f', f''.

Now define $\overline{\phi}, \overline{\psi}, \overline{g}, \overline{h}$ as follows.

Let $\overline{\phi}$ be the restriction of ϕ to the $\ker f'$. So given $x \in \ker f'$, we get $f'(x) = 0$. Implying that $hf'(x) = 0$. Now the commutativity of the appropriate square, we have $f\phi(x) = 0$. Giving $\phi(x) \in \ker f$. So ϕ sends elements of $\ker f'$ to $\ker f$, making $\overline{\phi}$ a well defined module homomorphism and the square

$$\begin{array}{ccc} \ker f' & \xrightarrow{\overline{\phi}} & \ker f \\ \downarrow{\scriptstyle i_{f'}} & & \downarrow{\scriptstyle i_f} \\ M' & \xrightarrow[\phi]{} & M \end{array}$$

commutes. A similar argument can be made for the square

$$\begin{array}{ccc} \ker f & \xrightarrow{\overline{g}} & \ker f'' \\ {\scriptstyle i_f}\downarrow & & \downarrow{\scriptstyle i_{f''}} \\ M & \xrightarrow[g]{} & M'' \end{array}$$

where $\overline{g}$ is the restriction of g to $\ker f$.

Note: That the module homomorphisms; $\overline{\phi}$ and $\overline{g}$ are the unique maps induced by $\ker f$ and $\ker f''$ respectively, from Theorem 2.14.

Now define $\overline{h} : \operatorname{cok} f' \longrightarrow \operatorname{cok} f$ as $\overline{h}(\overline{a}) = \pi_f h(a)$, and let $\overline{a} = \overline{b}$ be in the $\operatorname{cok} f'$. This implies that $a - b$ is in $\operatorname{Im} f'$, so for some $m \in M'$, $f'(m) = a - b$. Thus $h(a - b) = hf'(m)$ and, by the known commutativity, $h(a) - h(b) = f\phi(m)$. Implying that $h(a) - h(b)$ is in $\operatorname{Im} f$. Giving $\overline{h(a)} = \overline{h(b)}$ with $\overline{h(a)} = \pi_f h(a)$. Therefore, $\pi_f h(a) = \pi_f h(b)$ and $\overline{h}$ is well defined. Also, $\overline{h}(\overline{a}) = \pi_f h(a)$, so $\overline{h}\pi_{f'} = \pi_f h$ and the square

$$\begin{array}{ccc} N' & \xrightarrow{h} & N \\ {\scriptstyle \pi_{f'}}\downarrow & & \downarrow{\scriptstyle \pi_f} \\ \operatorname{cok} f' & \xrightarrow[\overline{h}]{} & \operatorname{cok} f \end{array}$$

commutes. Again a similar argument can be made for the square

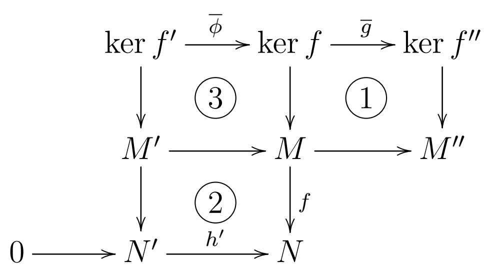
$$\begin{array}{ccc} N & \xrightarrow{\psi} & N'' \\ {\scriptstyle \pi_f}\downarrow & & \downarrow{\scriptstyle \pi_{f''}} \\ \operatorname{cok} f & \xrightarrow[\overline{\psi}]{} & \operatorname{cok} f'' \end{array}$$

Note: The module homomorphisms; $\overline{h}$ and $\overline{\psi}$ are the unique maps generated by $\operatorname{cok} f'$ and $\operatorname{cok} f''$ respectively, from Theorem 2.15.

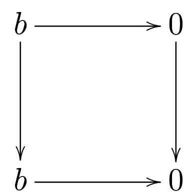
Thus $\overline{\phi}, \overline{\psi}, \overline{g}, \overline{h}$ are all well defined module homomorphisms and the induced diagram commutes. We will now show the exactness of each row.

We wish to show that $\operatorname{Im}\overline{\phi} \subseteq \ker \overline{g}$ which is equivalent to $\overline{g} \circ \overline{\phi} = 0$. Since $i_{f''}$ is injective it will suffice to show $i_{f''} \circ \overline{g} \circ \overline{\phi} = 0$. By the known commutativity, $i_{f''} \circ \overline{g} \circ \overline{\phi} = g \circ \phi \circ i_{f'}$. Since $g \circ \phi = 0$, we get $\overline{g} \circ \overline{\phi} = 0$, and $\operatorname{Im}\overline{\phi} \subseteq \ker \overline{g}$.

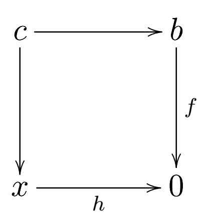
Now, suppose $b \in \ker f$ and $b \xrightarrow{\overline{g}} 0$, or that $b \in \ker \overline{g}$. From the following diagram



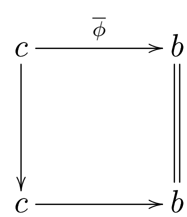
square ① will give



Which implies that $b \in \ker g$. Thus there exists a $c \in M'$ such that $c \longrightarrow b$. Now square ② gives



for some $x \in N'$. By commutativity and the fact that h is one-to-one, $x = 0$. So $c \in \ker f'$ and square ③ gives



Resulting in $b \in \operatorname{Im}\overline{\phi}$, so $\ker\overline{g} \subseteq \operatorname{Im}\overline{\phi}$ and hence $\ker\overline{g} = \operatorname{Im}\overline{\phi}$. Since $\overline{\phi}$ is defined as the restriction of ϕ to the kernel of f', we can see that $\overline{\phi}$ is one-to-one. Thus

$$0 \longrightarrow \ker f' \xrightarrow{\overline{\phi}} \ker f \xrightarrow{\overline{g}} \ker f''$$

is exact.

We now wish to show $\operatorname{Im}\overline{h} \subseteq \ker\overline{\psi}$, so let $\overline{a} \in \operatorname{cok} f'$. Thus $\overline{a} = \pi_f(a)$ for some $a \in N'$. $\psi \circ h = 0$ by exactness, and $\psi \circ h(a) = 0$ meaning $\pi_{f''}(\psi \circ h(a)) = \overline{0}$ in $\operatorname{cok} f''$. Then $\overline{\phi} \circ \overline{h}(\overline{a}) = \overline{\phi} \circ \overline{h} \circ \pi_f(a) = 0$ by commutativity, so $\overline{\psi} \circ \overline{h} = 0$, and $\operatorname{Im}\overline{h} \subseteq \ker\overline{\psi}$.

Now, let $\overline{x} \in \operatorname{cok} f$. This will have the form $\pi_f(x)$ for some $x \in N$. Suppose $\overline{x} \in \ker\overline{\psi}$. Using the following diagram

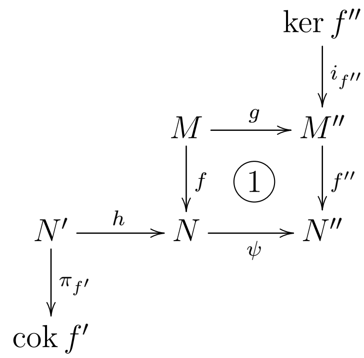
$$\begin{array}{ccccccc}
 & & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\
 & & \downarrow f & (1) & \downarrow f'' & & \\
N' & \xrightarrow{h} & N & \xrightarrow{\psi} & N'' & & \\
\downarrow \pi_{f'} & (2) & \downarrow \pi_f & & \downarrow \pi_{f''} & & \\
\operatorname{cok} f' & \xrightarrow{\overline{h}} & \operatorname{cok} f & \xrightarrow{\overline{\psi}} & \operatorname{cok} f'' & &
\end{array}$$

and the definition of $\overline{\psi}$ we get $\pi_{f''}\psi(x) = 0$. This implies that $\psi(x) \in \ker\pi_{f''} = \operatorname{Im} f''$. Thus for some $y \in M''$ we get $\psi(x) = f''(y)$. But g is onto, so there exists some $z \in M$ such that $g(z) = y$. By commutativity of (1), we find that $f''g(z) = \psi f(z)$ and thus $\psi(x) = \psi f(z)$. Giving $\psi(x - f(z)) = 0$ and $(x - f(z)) \in \ker\psi$. From exactness, there exists a unique w such that $h(w) = x - f(z)$. But again, by commutativity of (2), $\pi_f(x - f(z)) = \pi_f h(w) = \overline{h}\pi_{f'}(w) \in \operatorname{Im}\overline{h}$. However, $\pi_f \circ f = 0$, resulting in $\pi_f(x) = \overline{x} \in \operatorname{Im}\overline{h}$. Thus $\ker\overline{\psi} \subseteq \operatorname{Im}\overline{h}$, and hence $\ker\overline{\psi} = \operatorname{Im}\overline{h}$. Since $\overline{\psi}$ is defined as $\pi_{f''} \circ \psi$ and the composition of two onto functions is onto, we have $\overline{\psi}$ as onto and thus

$$\operatorname{cok} f' \xrightarrow{\overline{h}} \operatorname{cok} f \xrightarrow{\overline{\psi}} \operatorname{cok} f'' \longrightarrow 0$$

exact.

It remains to show that there exists a well defined module homomorphism, δ, completing the exact sequence. Let $z \in \ker f'' \subset M''$. From the following diagram



since g is surjective, there exists a $y \in M$ such that $g(y) = z$. By the commutativity of ①, $f''g(y) = 0 = \psi f(y)$ and by exactness $f(y) = h(x)$ for some $x \in N'$. Define $\delta(z) = \overline{x} \in \operatorname{cok} f'$.

Now suppose $y' \in M$ with $g(y') = i_{f''}(z)$. Then $(y - y') \in \ker g = \operatorname{Im} \phi$, so $y - y' = \phi(a)$ for some $a \in M'$, and by commutativity we get $f(y-y') = h \circ f'(a)$. From above we can see that $f(y) = h(x)$ and $f(y') = h(x')$ for some $x, x' \in N'$. Thus $(h \circ f')(a) = h(x - x')$ implying that $(x - x' - f'(a)) \in \ker h$. Since h is injective, $x - x' - f'(a) = 0$ or $x - x' = f'(a)$. Then from $\pi_{f'}$ we get $\overline{x - x'} = \overline{0}$ implying that $\overline{x} = \overline{x'}$. Giving δ as a well defined map.

Suppose that $a, b \in \ker f''$. By definition of δ, $\delta(a) = \overline{c}$ and $\delta(b) = \overline{d}$ where $g(u) = a$ and $f(u) = h(c)$ for some $u \in M$ and $c \in N'$ as well as $g(v) = b$ and $f(v) = h(d)$ for some $v \in M$ and $d \in N'$. Now

$$\begin{aligned} g(u + rv) &= g(u) + rg(v) \\ &= a + rb \end{aligned}$$

and

$$\begin{aligned} f(u+rv) &= f(u)+rf(v) \\ &= h(c)+rh(d) \\ &= h(c+rd) \end{aligned}$$

Thus

$$\begin{aligned} \delta(a+rb) &= \overline{c+rd} \\ &= \overline{c}+r\overline{d} \\ &= \delta(a)+r\delta(b) \end{aligned}$$

Resulting in δ a module homomorphism.

Finally, we need to show exactness involving δ. For $z = \overline{g}(a)$ where $a \in \ker f$, we get $z = g(a)$ by definition of $\overline{g}$ and $f(a) = 0$. Then $f(a) = 0 = h(0)$. Hence, by definition of δ, we get $\delta\overline{g}(a) = \overline{0} = 0$. Thus $\delta \circ \overline{g} = 0$ and $\operatorname{Im} \overline{g} \subseteq \ker \delta$.

Let $z \in \ker \delta$, so $\delta(z) = \overline{0}$. Thus by the definition of δ, there exists a $y \in M$ such that $g(y) = z$ and $f(y) = h(x)$ where $\delta(z) = \overline{x}$. Hence $\overline{x} = \overline{0}$ and $x \in \operatorname{Im} f'$, so there exists a $u \in M'$ where $f'(u) = x$. Resulting in $h(x) = hf'(u)$. By commutativity, we get $hf'(u) = f\phi(u)$. Replacing y with $y - \phi(u)$ we have

$$\begin{aligned} g(y-\phi(u)) &= g(y)-g\phi(u) \\ &= z-0 \\ &= z \end{aligned}$$

and $f(y-\phi(u)) = 0$, so $(y-\phi(u)) \in \ker f$ and thus

$$\begin{aligned} \overline{g}(y-\phi(u)) &= g(y-\phi(u)) \\ &= z \end{aligned}$$

Resulting in $z \in \operatorname{Im} \overline{g}$, giving $\ker \delta \subseteq \operatorname{Im} \overline{g}$. Hence $\operatorname{Im} \overline{g} = \ker \delta$.

Now for $z \in \ker f''$, we have $\delta(z) = \overline{x}$ where $g(y) = z$ and $f(y) = h(x)$, by definition of δ. Thus

$$\begin{aligned} \overline{h}\delta &= \overline{h}(\overline{x}) \\ &= \overline{h(x)} \\ &= \overline{f(y)} \\ &= 0 \end{aligned}$$

since $f(y) \in \operatorname{Im} f$. Resulting in $\overline{h} \circ \delta = 0$ and $\operatorname{Im} \delta \subseteq \ker \overline{h}$.

Finally, let $\overline{x} \in \ker \overline{h}$, so $\overline{0} = \overline{h}(\overline{x}) = \overline{h(x)}$, thus $h(x) \in \operatorname{Im} f$ and $h(x) = f(y)$ for some $y \in M$. Let $g(y) = z \in M''$. Since

$$\begin{array}{ccc} M & \xrightarrow{g} & M'' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\ N & \longrightarrow & N'' \end{array}$$

commutes, we can create the square

$$\begin{array}{ccc} y & \xrightarrow{g} & z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\ h(x) & \longrightarrow & 0 \end{array}$$

Thus $f''(z) = (\psi \circ h)(x) = 0$, so $z \in \ker f''$ with $g(y) = z$ and $h(x) = f(y)$. Thus by definition, $\delta(z) = \overline{x}$. Giving $\ker \overline{h} \subseteq \operatorname{Im} \delta$. Thus $\ker \overline{h} = \operatorname{Im} \delta$, completing the proof of the Snake Lemma. ∎

Lemma 2.19 Let $B \rightarrowtail E \twoheadrightarrow A$ and $B' \rightarrowtail E' \twoheadrightarrow A'$ be two short exact sequences where in the commutative diagram:

$$\begin{CD} B @>\mu>> E @>\nu>> A \\ @V\alpha VV @V\beta VV @V\gamma VV \\ B' @>>\mu'> E' @>>\nu'> A' \end{CD}$$

any of the two homomorphism α, β, or γ are isomorphisms. Then the third is an isomorphism.

Proof:

By the Snake Lemma, there exists the connecting homomorphism δ and an induced digram

$$\begin{CD} @. 0 @. 0 @. 0 \\ @. @VVV @VVV @VVV \\ @. K_1 @>>> K_2 @>>> K_3 \\ @. @VVV @VVV @VVV \\ @. B @>>> E @>>> A @>>> 0 \\ @. @V\alpha VV @V\beta VV @V\gamma VV \\ 0 @>>> B' @>>> E' @>>> A' \\ @. @VVV @VVV @VVV \\ @. C_1 @>>> C_2 @>>> C_3 \\ @. @VVV @VVV @VVV \\ @. 0 @. 0 @. 0 \end{CD}$$

such that the sequence

$$0 \longrightarrow K_1 \longrightarrow K_2 \longrightarrow K_3 \xrightarrow{\delta} C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow 0$$

is exact.

Case 1: Now suppose that α and β are isomorphisms. This gives $K_1 = K_2 = C_1 = C_2 = 0$. Giving the exact sequence

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow K_3 \longrightarrow 0 \longrightarrow 0 \longrightarrow C_3 \longrightarrow 0$$

Since this sequence is exact, $K_3 = C_3 = 0$. $K_3 = 0$ if and only if γ is injective and $C_3 = 0$ if and only if γ is surjective. Thus γ is an isomorphism.

Case 2: Now suppose that α and γ are isomorphisms. This gives $K_1 = K_3 = C_1 = C_3 = 0$. Giving the exact sequence

$$0 \longrightarrow 0 \longrightarrow K_2 \longrightarrow 0 \longrightarrow 0 \longrightarrow C_2 \longrightarrow 0 \longrightarrow 0$$

Since this sequence is exact, $K_2 = C_2 = 0$. Implying, by the same reasoning as above that β is an isomorphism.

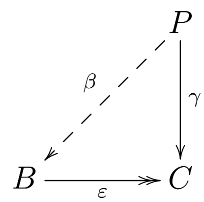
Case 3: Now suppose that β and γ are isomorphisms. This gives $K_2 = K_3 = C_2 = C_3 = 0$. Giving the exact sequence

$$0 \longrightarrow K_1 \longrightarrow 0 \longrightarrow 0 \longrightarrow C_1 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

Since this sequence is exact, $K_1 = C_1 = 0$. Finally implying that α is an isomorphism. ∎

2.4 Projective and Injective Modules

Definition 2.20 A R-module P is projective if for every onto morphism $\varepsilon : B \to C$ and every homomorphism $\gamma : P \to C$ there exists a module homomorphism $\beta : P \to B$ such that $\varepsilon\beta = \gamma$. Illustrated by the following commutative diagram:



◊

Definition 2.21 A *free module* is a module with a basis. Where for an R-module M, the set $E \subseteq M$ is a basis for M if for $r_i \in R$, $e_i \in E$ all i

(i) for any $m \in M$, $m = r_1e_1 + r_2e_2 + \cdots + r_ne_n$ for some $n \in \mathbb{N}$ and

(ii) E is independent, that is, $r_1e_1 + r_2e_2 + \cdots + r_ne_n = 0_M$ where $e_i \neq e_j$ for $i \neq j$ if and only if $r_1 = r_2 = \cdots = r_n = 0_R$.

◊

Theorem 2.22 Every module M is a quotient of a free module.

Proof: Let F be the free module generated by the set X. Where X is the set of all non-zero elements of M. Then the inclusion $i : X \longrightarrow M$ induces an epimorphism $f : F \longrightarrow M$ defined by $f(\Sigma r_i(x_i)) = \Sigma r_i x_i$, for $r_i \in R$ and $x_i \in X$ for all i. Then by the first isomorphism theorem, $M \cong F/\ker f$. ∎

Let F be a free module and $\{y_i : i \in I\}$ be a basis for F. Suppose that $\gamma : F \to C$ and $\varepsilon : B \to C$ are module homomorphisms with ε onto. So for every $i \in I$ there exists some $x_i \in B$ such that $\varepsilon(x_i) = \gamma(y_i)$. Now define $\beta : F \to B$ such that $\beta(\Sigma(r_i y_i)) = \Sigma r_i x_i$. Since $\{y_i : i \in I\}$ forms a basis it follows that β is a well defined homomorphism. So

$$\begin{aligned}\varepsilon\beta(\Sigma r_i y_i) &= \varepsilon(\Sigma r_i x_i)\\ &= \Sigma r_i \varepsilon(x_i)\\ &= \Sigma r_i \gamma(y_i)\\ &= \gamma(\Sigma r_i y_i)\end{aligned}$$

Thus $\gamma = \varepsilon\beta$ and hence F is projective. Meaning that free modules are projective modules.

Definition 2.23 The short exact sequence,

$$0 \longrightarrow A \xrightarrow{\alpha} E \xrightarrow{\beta} B \longrightarrow 0$$

is said to split if there exists a pair of homomorphisms, $\hat{\alpha}$ and $\hat{\beta}$, such that $\hat{\alpha}\alpha = 1_A$ and $\beta\hat{\beta} = 1_B$. ♢

Example 2.24 Consider the sequence

$$0 \longrightarrow A \xrightarrow{i_A} A \oplus B \xrightarrow{\pi_B} B \longrightarrow 0,$$

where $i_A : A \longrightarrow A \oplus B$ is defined by $i_A(a) = (a, 0)$, which is clearly one-to-one, and $\pi_B : A \oplus B \longrightarrow B$ is defined by $\pi(a, b) = b$, which is clearly onto. Thus $\operatorname{Im} i_A = \ker \pi_B$, giving a short exact sequence.

We can the reverse the arrows as follows:

$$0 \longrightarrow B \xrightarrow[i_B]{} A \oplus B \xrightarrow[\pi_A]{} A \longrightarrow 0.$$

This would then give $\pi_A i_A = 1_A$ and $i_B \pi_B = 1_B$. Hence the sequence is a splitting sequence.

Note: The composition of these functions cannot always be reversed. $\hat{\alpha}$ acts strictly as a left inverse and $\hat{\beta}$ acts as a right inverse. We then call $\hat{\alpha}$ and $\hat{\beta}$ splitting morphisms.

Our goal from here is to show for a split short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

that $E \cong A \oplus B$. First we will consider the following lemma.

Lemma 2.25 Given a short exact sequence,

$$0 \longrightarrow A \xrightarrow{\alpha} E \xrightarrow{\beta} B \longrightarrow 0,$$

the following are equivalent.

(i) There exists a module homomorphism, $\hat{\alpha}$, with $\hat{\alpha}\alpha = 1_A$;

(ii) There exists a module homomorphism, $\hat{\beta}$, with $\beta\hat{\beta} = 1_B$;

(iii) There exists $\hat{\alpha}$ and $\hat{\beta}$ with $\alpha\hat{\alpha} + \hat{\beta}\beta = 1_E$,

(iv) The sequence splits.

Proof: Assume there exists $\hat{\alpha}$ such that $\hat{\alpha}\alpha = 1_A$ and consider the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & B & \longrightarrow & 0 \\ & & & & \big\downarrow{\scriptstyle 1_E - \alpha\hat{\alpha}} & & & & \\ & & & & E & & & & \end{array}$$

Then

$$\begin{aligned} (1_E - \alpha\hat{\alpha})\alpha &= 1_E\alpha - \alpha\hat{\alpha}\alpha \\ &= \alpha - \alpha 1_A \\ &= \alpha - \alpha \\ &= 0 \end{aligned}$$

so by Theorem 2.15 we know there exists a $\hat{\beta} : B \longrightarrow E$ such that $\hat{\beta}\beta = 1_E - \alpha\hat{\alpha}$. Thus $1_E = \alpha\hat{\alpha} + \hat{\beta}\beta$. Now assume there exists $\hat{\beta}$ such that $\beta\hat{\beta} = 1_B$ and consider the following diagram.

$$\begin{array}{ccccccccc} & & & & E & & & & \\ & & & & \downarrow{\scriptstyle 1_E - \hat{\beta}\beta} & & & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & B & \longrightarrow & 0 \end{array}$$

Then

$$\begin{aligned} \beta(1_E - \hat{\beta}\beta) &= \beta 1_E - \beta\hat{\beta}\beta \\ &= \beta - 1_B\beta \\ &= \beta - \beta \\ &= 0 \end{aligned}$$

so by Theorem 2.14 we know there exits a $\hat{\alpha} : E \longrightarrow A$ such that $\alpha\hat{\alpha} = 1_E - \hat{\beta}\beta$. Thus $1_E = \alpha\hat{\alpha} + \hat{\beta}\beta$. Resulting in $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (iii)$.
Now assume (iii) and apply α to $1_E = \alpha\hat{\alpha} + \hat{\beta}\beta$ to get

$$\begin{aligned} 1_E\alpha &= (\alpha\hat{\alpha} + \hat{\beta}\beta)\alpha \\ &= \alpha\hat{\alpha}\alpha + \hat{\beta}\beta\alpha \end{aligned}$$

Exactness would then imply that $\beta\alpha = 0$ and thus $1_E\alpha = \alpha\hat{\alpha}\alpha$. However, this gives us that $1_E\alpha = \alpha 1_A = \alpha(\hat{\alpha}\alpha)$. Since α is a monomorphism, $1_A = \hat{\alpha}\alpha$. A similar argument can be made for $\beta\hat{\beta} = 1_B$. Thus $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$. Giving $(i) \Leftrightarrow (iii) \Leftrightarrow (ii)$. Consequently, $(i) \Leftrightarrow (iv) \Leftrightarrow (ii)$ by definition of splitting sequences. Thus concluding the proof. ∎

We then prove the following theorems simultaneously.

Theorem 2.26 If the short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} E \xrightarrow{\beta} B \longrightarrow 0$$

splits, then the sequence

$$0 \longrightarrow B \xrightarrow{\hat{\beta}} E \xrightarrow{\hat{\alpha}} A \longrightarrow 0$$

is also a short exact splitting sequence.

Theorem 2.27 If the short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} E \xrightarrow{\beta} B \longrightarrow 0$$

splits. Then

$$\begin{aligned} E &= \operatorname{Im}\alpha \oplus \operatorname{Im}\hat{\beta} \\ &= \operatorname{Im}\alpha \oplus \ker\hat{\alpha} \\ &= \ker\beta \oplus \operatorname{Im}\hat{\beta} \end{aligned}$$

Hence, $E \cong A \oplus B$.

Proof: Given $e \in E$, $1_E = \alpha\hat{\alpha} + \hat{\beta}\beta$, from Lemma 2.25, thus $e = \alpha\hat{\alpha}e + \hat{\beta}\beta e$. Since $\alpha\hat{\alpha}e \in \operatorname{Im}\alpha$ and $\hat{\beta}\beta e \in \operatorname{Im}\hat{\beta}$, $E = \operatorname{Im}\alpha + \operatorname{Im}\hat{\beta}$. Now assume $e \in \operatorname{Im}\alpha \cap \operatorname{Im}\hat{\beta}$. This would then imply that $e = \alpha a$, for some $a \in A$, and $e = \hat{\beta}b$, for some $b \in B$. Resulting in $\alpha a = \hat{\beta}b$. Applying β to each side we have $\beta\alpha a = \beta\hat{\beta}b$. Since the sequence was a short exact splitting sequence, we know that $\beta\alpha = 0$ and $\beta\hat{\beta} = 1_B$, thus $0 = b$ and then $e = \hat{\beta}b = 0$ as well giving $\operatorname{Im}\alpha \cap \operatorname{Im}\hat{\beta} = 0$. Hence $E = \operatorname{Im}\alpha \oplus \operatorname{Im}\hat{\beta}$. To show $\ker\hat{\alpha} = \operatorname{Im}\hat{\beta}$, suppose that $e \in \ker\hat{\alpha}$,

$$\begin{aligned} e &= 1_E e \\ &= \alpha\hat{\alpha}e + \hat{\beta}\beta e \\ &= \hat{\beta}\beta e \end{aligned}$$

Thus $\ker\hat{\alpha} \subseteq \operatorname{Im}\hat{\beta}$. Now conversely, suppose that $e = \hat{\beta}b$. Giving

$$\begin{aligned}
\hat{\beta}b = &\ 1_E\hat{\beta}b \\
= &\ \alpha\hat{\alpha}\hat{\beta}b + \hat{\beta}\beta\hat{\beta}b && \text{Lemma 2.25:}(iii) \\
= &\ \alpha\hat{\alpha}\hat{\beta}b + \hat{\beta}b && \text{Lemma 2.25:}(ii) \\
0 = &\ \alpha\hat{\alpha}\hat{\beta}b && \hat{\beta}b - \hat{\beta}b = 0 \\
0 = &\ \alpha\hat{\alpha}e
\end{aligned}$$

Since α is a monomorphism, we find that $0 = \hat{\alpha}e$. Resulting in $\operatorname{Im}\hat{\beta} \subseteq \ker\hat{\alpha}$ and hence $\operatorname{Im}\hat{\beta} = \ker\hat{\alpha}$.

Now $\hat{\alpha}\alpha = 1_A$ would imply that $\hat{\alpha}$ is an epimorphism and $\beta\hat{\beta} = 1_B$ would imply that $\hat{\beta}$ is a monomorphism. Therefore

$$0 \longrightarrow B \xrightarrow{\hat{\beta}} E \xrightarrow{\hat{\alpha}} A \longrightarrow 0$$

is a short exact splitting sequence.

Finally, α induces an isomorphism $A \longrightarrow \operatorname{Im}\alpha$ and $\hat{\beta}$ induces an isomorphism $B \longrightarrow \operatorname{Im}\hat{\beta}$. We then get $E \cong A \oplus B$, thus completing the proofs.

∎

Theorem 2.28 An R-module P is projective if and only if any exact sequence of the form

$$0 \longrightarrow A \xrightarrow{\psi} E \xrightarrow{\phi} P \longrightarrow 0$$

splits.

We will first prove the following lemma to aid in the proof of this theorem.

Lemma 2.29 Let A and B be R-modules. If the external direct sum $A \oplus B$ is free then, A and B are projective.

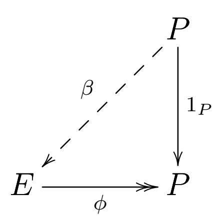
Proof: Let $F = A \oplus B$. By symmetry, we will only need to prove that A is projective. Define the maps $\alpha : A \to F$ and $\beta : F \to A$ by $\alpha(x) = (x, 0)$, the injection, and $\beta(x, y) = x$, the projection, for all $x \in A$ and $y \in B$. Note that $\beta\alpha = 1_A$. Suppose that we have R-module homomorphisms $h : A \to N$ and $f : M \to N$, where f is onto. So we have an R-module homomorphism $h\beta : F \to N$ and thus, since F is projective, there exists an R-module homomorphism $\gamma : F \to M$ such that $f\gamma = h\beta$. Let $g = \gamma\alpha$. Then $fg = f\gamma\alpha = h\beta\alpha = h$, and thus A is projective. ∎

We will now prove Theorem 2.28.

Proof: Assume P is projective and

$$0 \longrightarrow A \xrightarrow{\psi} E \xrightarrow{\phi} P \longrightarrow 0$$

is a short exact sequence. Since ϕ is onto and P is projective, there exists a β such that



commutes. Hence $\phi\beta = 1_P$ and by Lemma 2.25 the sequence splits. Now assume every short exact sequence

$$0 \longrightarrow A \xrightarrow{\psi} E \xrightarrow{\phi} P \longrightarrow 0$$

splits. Since every R-module is a quotient of a free module by Theorem 2.22, we have a free module F and an onto R-module homomorphism $f_2 : F \to P$. Let $K = \ker f_2$. Thus we have the exact sequence

$$0 \longrightarrow K \xrightarrow{f_1} F \xrightarrow{f_2} P \longrightarrow 0$$

which, by assumption, splits. Thus $F \cong P \oplus K$. Implying that $P \oplus K$ is free and by Lemma 2.29, P is projective. ∎

Example 2.30 For a ring R with $e \in R$ an idempotent ($e^2 = e$), we would like to show $R = Re \oplus R(1-e)$. Since for any $r \in R$ $r = re + r(1-e)$, we get that $R = Re + R(1-e)$. Now if $z \in Re \cap R(1-e)$, then $z = ve$ for some $v \in R$ and $z = w(1-e)$ for some $w \in R$. From $z = ve$ we get

$$\begin{aligned} z &= ve \\ ze &= vee \\ &= ve^2 \\ &= ve \\ &= z \end{aligned}$$

Now from $z = w(1-e)$

$$\begin{aligned} z &= w(1-e) \\ ze &= w(1-e)e \\ &= w(e-e^2) \\ &= w(e-e) \\ &= 0 \end{aligned}$$

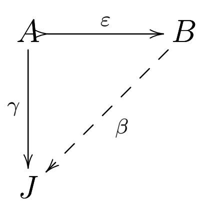
Which results in $z = ze = 0$ and thus $Re \cap R(1-e) = \emptyset$. Making $R = Re \oplus R(1-e)$ a free R-module with basis $E = \{1_R\}$ and thus Re and $R(1-e)$ are both projective.

Now for

$$R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$$

where F is a field let $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Thus $Re = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$. As a vector space over F, R will be dimension 3. However, for any free module with basis of size n, we would have a vector space of dimension $3n$ over F. But Re is one dimensional; meaning that Re is not free. However, it is still projective, so the concept of projective is more general than free.

Definition 2.31 A R-module J is injective if for every one-to-one morphism $\varepsilon : A \to B$ and to every homomorphism $\gamma : A \to J$ there exists a module homomorphism $\beta : B \to J$ such that $\beta\varepsilon = \gamma$. Illustrated by the following commutative diagram:



◊

Example 2.32 ℚ*, the rational numbers, is an injective* ℤ*-module.*[2, p. 41]

Chapter 3

Categories

To be able to discuss the ideas of homology it is also necessary for the reader to be familiar with the concepts of categories. Categories give us the ability to discuss more general ideas. For example, one could not talk about the set of all sets without running into a problem with Russel's Paradox. The following sections will be focused on the ideas and topics necessary for this thesis and should not be viewed as an overview of the subject of categories.

Definition 3.1 A *category* $\mathbf{C}$ consists of a class of *objects* and sets of *morphisms* between those objects. For every ordered pair A, B of objects there is a set $Hom_{\mathbf{C}}(A, B)$ of morphisms from A to B, and for every ordered triple A, B, C of objects there is a *law of composition* of morphisms, a map

$$Hom_{\mathbf{C}}(A, B) \times Hom_{\mathbf{C}}(B, C) \longrightarrow Hom_{\mathbf{C}}(A, C)$$

where $(f, g) \mapsto gf$, and gf is called the composition of g with f. The objects and morphisms must satisfy the following axioms: for objects A, B, C, and D:

(i) if $A \neq C$ or $B \neq D$, the $Hom_{\mathbf{C}}(A, B)$ and $Hom_{\mathbf{C}}(C, D)$ are disjoint sets,

(ii) composition of morphisms is associative $h(gf) = (hg)f$ for every $f \in Hom_{\mathbf{C}}(A, B), g \in Hom_{\mathbf{C}}(B, C)$, and $h \in Hom_{\mathbf{C}}(C, D)$),

(iii) each object has an identity morphism, i.e., for every object A there is a morphism $1_A \in Hom_{\mathbf{C}}(A, A)$ such that $f1_A = f$ for every $f \in Hom_{\mathbf{C}}(A, B)$ as well as $1_A g = g$ for every $g \in Hom_{\mathbf{C}}(C, A)$.

$\diamondsuit$

When discussing the idea of $Hom_{\mathbf{C}}(-,-)$, we shall suppress the subscript when it is clear which category the objects are coming from. Also, a morphism from objects A to B will be denoted as $f : A \longrightarrow B$ or $A \xrightarrow{f} B$. By definition, A is the domain of the morphism while B is the codomain. An isomorphism f is a morphism $f : A \longrightarrow B$ if there exists a morphism $g : B \longrightarrow A$ such that $gf = 1_A$ and $fg = 1_B$.

Naturally one can extend this to the idea of subcategories. A *subcategory* $\mathbf{C}$ of a category $\mathbf{D}$ occurs when every object of $\mathbf{C}$ is also an object of $\mathbf{D}$. Also, given $A, B \in \mathbf{C}$ we have $Hom_{\mathbf{C}}(A, B) \subseteq Hom_{\mathbf{D}}(A, B)$. Furthermore $\mathbf{C}$ is called a *full subcategory* when $Hom_C(A, B) = Hom_D(A, B)$ for all objects A and B in $\mathbf{D}$.

Example 3.2 Set is the category of all sets. Given two sets A and B, $Hom(A, B)$ is the set of all functions from A to B. Composition of morphisms works the same as composition of functions and the identity in $Hom(A, A)$ is the identity function on the set A. This category contains the category of all finite sets as a full subcategory.

Grp is the category of all groups. Here the objects are groups and morphisms are group homomorphisms. Since the composition of group homomorphims is also group homomorphism all of the axioms are satisfied. Similarly, to **Set** the category of all abelian groups, **Ab** is a full subcategory of **Grp**. **Ring** is also a category. The objects of **Ring** are nonzero rings with 1 and morphisms are ring homomorphisms that send 1 to 1.

Given a fixed ring R, the category R-**Mod** consists of all left modules with morphisms being R-module homomorphisms. □

Example 3.3 Consider R a ring with unity, we will define a category as such:

- R is the only object,
- morphisms are the elements of R.

The identity morphism is the multiplicative identity and composition of a and b is defined by multiplication of a and b. Also, for $a, b, c \in R$ we have $(ab)c = a(bc)$ by definition of a ring. Thus we satisfy all of the axioms of a category. We will denote this category by $\mathbf{R} - \mathbf{cat}$. □

Example 3.4 *Directed Multigraphs*

Here we introduce the concept of directed graphs. Directed graphs (or digraphs) will act as an underlying example of the theory presented throughout this thesis.

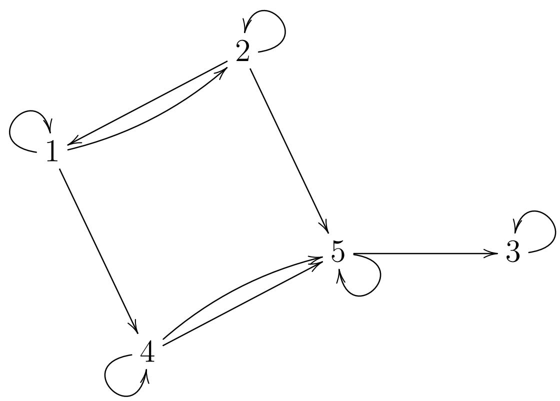
Definition 3.5 A directed graph is an ordered pair, $G = (V, E)$, where

- V is a set denoting the vertices,
- $E \subseteq V \times V$ is a set of ordered pairs denoting the edges.

$\diamondsuit$

Each edge can be denoted as $e = (x, y)$ as the edge from vertex x to vertex y. When using multiple edges from one vertex to another we will label them using subscripts. For the multiple edges from x to y, these will be denoted by $(x,y)_1, (x,y)_2, ..., (x,y)_i$. A path is a finite string of consecutive edges. Define the empty path as the edge (x,x), an edge that goes back to itself. So for example: $V = \{1,2,3,4,5\}$

$E = \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,2),(2,1),(1,4),(4,5)_1,(4,5)_2,(2,5),(5,3)\}$



□

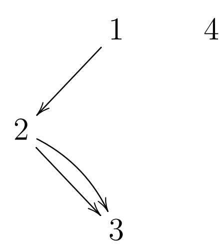
Given a digraph, we can create a new *completed* digraph. Paths in the original digraph are redrawn via concatenation as follows: $(1,2)||(2,3) = (1,3)$.

$$1 \longrightarrow 2 \longrightarrow 3$$

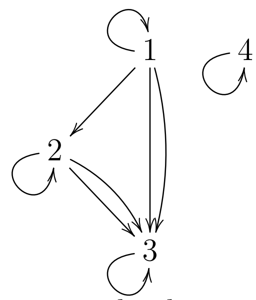
becomes the edge

$$1 \longrightarrow 3$$

The empty path then becomes the empty edge. For example the graph G



becomes the completed graph



This new digraph is drawn in exactly the same way as the original digraph. We can then create these completed graphs.

Each of these completed digraphs forms a category where the objects are the vertices, and the morphisms are the paths. The empty path acts as the identity on each object. This category is known as the Quiver of G. Denoted as **Quiv**(G).

3.1 Functor

Given two categories **C** and **D** it is natural for one to ask if there is some sort of way to change from one category to another.

Example 3.6 Given the category **Grp** one could change it to a subcategory of **Set** by forgetting the group structure of these objects. In other words, forgetting that all of these objects are groups and morphisms are homomorphisms. This is called a *forgetful functor*. □

Definition 3.7 Let **C** and **D** be categories. A *covariant functor*, $\mathbf{F} : \mathbf{C} \to \mathbf{D}$ is a mapping that

- associates to each object $X \in \mathbf{C}$ an object $\mathbf{F}(X) \in \mathbf{D}$

- associates to each morphism $f : X \to Y$ in $\mathbf{C}$ to a morphism $\mathbf{F}(f) : \mathbf{F}(X) \to \mathbf{F}(Y)$ in $\mathbf{D}$ such that

  (1) $\mathbf{F}(1_X) = 1_{\mathbf{F}(X)}$ for every object $X \in \mathbf{C}$

  (2) $\mathbf{F}(f \circ g) = \mathbf{F}(f) \circ \mathbf{F}(g)$ for all morphisms $g : X \to Y$ and $f : Y \to Z$ in $\mathbf{C}$.

$\diamondsuit$

In other words, a generic functor is a map that preserves identity morphisms and composition of morphisms. When clear, we will drop the parenthesis and bold facing on the functor so that $\mathbf{F}(f) = Ff$ and $\mathbf{F}(X) = FX$. The idea of a *contravariant functor* is the same; however, the composition is reversed in axiom (2), $F(f \circ g) = Fg \circ Ff$.

Example 3.8 $Hom_R(M, -)$

We saw earlier, Hom sets for R-modules have an abelian structure. So in actuality Hom can act as a functor from $R - \mathbf{Mod}$ into $\mathbf{Ab}$.

Taking a closer look at the functor
$\mathbf{F} = Hom_R(M, -) : R - \mathbf{Mod} \to \mathbf{Ab}$. Each object N will be sent to an abelian group $Hom_R(M, N)$. Now given a module homomorphism, $\psi : N \to X$, $\mathbf{F}\psi$ will send ψ to a corresponding group homomorphism $\psi^*(\phi) = \psi\phi$. Given two composeable module homomorphisms

$$N \xrightarrow{\psi_1} X \xrightarrow{\psi_2} Y$$

we see that

$$\begin{aligned}
\mathbf{F}\psi_2\psi_1(\phi) &= \psi_2^*\psi_1^*(\phi) \\
&= \psi_2\psi_1\phi \\
&= \psi_2(\psi_1\phi) \\
&= \psi_2(\psi_1^*(\phi)) \\
&= \psi_2^*(\psi_1^*(\phi)) \\
&= (\mathbf{F}\psi_2)(\mathbf{F}\psi_1)\phi
\end{aligned}$$

Giving $\mathbf{F}\psi_2\psi_1 = \mathbf{F}\psi_2\mathbf{F}\psi_1$. Now for $\psi = I_N$, where I_N is the identity homomorphism on the module N, $I_N^* = I_N\phi = \phi$ for all $\phi \in Hom(M, N)$, so I_N^* is the identity on $Hom(M, N)$. Thus $\mathbf{F}$ is a covariant functor.

Considering $\mathbf{G} = Hom_R(-, N)$. Each module M is sent to a corresponding abelian group $Hom(M, N)$, and each module homomorphism $\varepsilon : M \to X$ would be sent to ε_* such that $\varepsilon_*(\phi) = \phi\varepsilon$. Here we have $\mathbf{G}\psi_2\psi_1 = (\mathbf{G}\psi_1)(\mathbf{G}\psi_2)$. Making $Hom(-, N)$ and contravariant functor. This leads to the following definition.

Definition 3.9 Given two categories $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{F}(-,-)$ is a *bifunctor* if it is a functor in both arguments with appropriate variances. Furthermore, if $\mathbf{F}$ is covariant in all arguments, then we require for morphisms $A \to A'$ and $B \to B'$ the following commutative diagram.

$$\begin{array}{ccc}
\mathbf{F}(A, B) & \longrightarrow & \mathbf{F}(A, B') \\
\downarrow & & \downarrow \\
\mathbf{F}(A', B) & \longrightarrow & \mathbf{F}(A', B')
\end{array}$$

Reversing the appropriate arrows for contravariance when necessary. $\diamondsuit$

$Hom(-,-)$ as a bifunctor will give us the following commutative diagram.

$$\begin{array}{ccc} Hom(M,N) & \longrightarrow & Hom(M,N') \\ \uparrow & & \uparrow \\ Hom(M',N) & \longrightarrow & Hom(M',N') \end{array}$$

Example 3.10 Let G be a digraph. Define the functor from the category of the quiver of G into the category of vector spaces over a field F,

$\mathbf{F} : \mathbf{Quiv}(G) \longrightarrow \mathbf{Vec}F$, as follows:

- each vertice i is sent to a vector space over F, V_i,
- each edge $e_j = (i_k, i_l)$ is sent to a linear transformation $L_j : V_k \to V_l$.

To check that composition is preserved, consider the sub-digraph

$$1 \longrightarrow 2 \longrightarrow 3$$

$\mathbf{F}$ will then map this to the following

$$V_1 \xrightarrow{L_1} V_2 \xrightarrow{L_2} V_3$$

With the concatenation of $e_1 = (1,2)$ and $e_2 = (2,3)$ we get

$$\mathbf{F}(e_1||e_2) = \mathbf{F}((1,3)) \quad (3.1)$$

$$= L_3 \quad (3.2)$$

Since $\mathbf{Quiv}(G)$ is created from the completed graph of G, we then can define $L_3 = L_2L_1$ since the composition of linear transformations is itself a linear transformation. Thus

$$= L_2L_1 \quad (3.3)$$

$$= \mathbf{F}(e_2)\mathbf{F}(e_1) \quad (3.4)$$

Meaning composition is preserved. We can map each empty path to I_{V_i} preserving identities. Thus confirming that $\mathbf{F}$ is a functor.

Here $\mathbf{F}(e_1||e_2) = \mathbf{F}e_1\mathbf{F}e_2$ can represent the "steps" a linear transformation at each stage illustrated on a digraph, much like a flow chart. Such functors are called the vector space representations of G and are denoted as $\mathbf{Rep}_F(G)$. □

Definition 3.11 For categories $\mathbf{C}$ and $\mathbf{D}$ and a functor $\mathbf{F} : \mathbf{C} \longrightarrow \mathbf{D}$ with $Hom_{\mathbf{C}}(C_1, C_2)$ and $Hom_{\mathbf{D}}(D_1, D_2)$ abelian groups for any C_1, C_2 in $\mathbf{C}$ and any D_1, D_2 in $\mathbf{D}$, then $\mathbf{F}$ is called *an additive functor* if $\mathbf{F}(\phi_1+\phi_2) = \mathbf{F}(\phi_1)+\mathbf{F}(\phi_2)$ for $\phi_1, \phi_2 \in Hom_{\mathbf{C}}(C_1, C_2)$.

In other words, $\mathbf{F}$ acts as an abelian group homomorphism, $\mathbf{F} : Hom(C_1, C_2) \longrightarrow Hom(\mathbf{F}C_1, \mathbf{F}C_2)$. ♢

Example 3.12 *With the ring R,* **R-cat** *into abelian groups*
Given an additive functor $\mathbf{F}$:**R-cat** $\longrightarrow$ **Ab** then:

- $\mathbf{F}R = M$, where M is an abelian group,
- for $a \in R$, set $\mathbf{F}a = l_a$, then l_a is an additive homomorphism $M \longrightarrow M$.

Here $\mathbf{F}(ab) = \mathbf{F}a\mathbf{F}b$ means that we can take any $m \in M$ and find how b acts on m then how a acts on that, or we may simply find how ab acts on m. So we have an action of R on the abelian group M. Thus for $a \in R$ and $m \in M$, we can define

$$am = l_a(m) \tag{3.5}$$

Given $a, b \in R$ and $m, n \in M$ we have

$$\begin{aligned}
(a+b)m &= \mathbf{F}(a+b)(m) \\
&= (\mathbf{F}a + \mathbf{F}b)(m) \\
&= \mathbf{F}a(m) + \mathbf{F}b(m) \\
&= am + bm \\
(ab)m &= \mathbf{F}(ab)m \\
&= \mathbf{F}a(bm) \\
&= \mathbf{F}a(\mathbf{F}b(m)) \\
&= a(bm) \\
a(m+n) &= \mathbf{F}a(m+n) \\
&= \mathbf{F}(am+an) \\
&= \mathbf{F}a(m) + \mathbf{F}a(n) \\
&= am + an
\end{aligned}$$

Finally, for $1 \in R$ we get $1m = \mathbf{F}1(m) = m$. Thus satisfying all the axioms of a left R-modules.

Conversely, given any left R-module M, we can define a functor $\mathbf{F}_M$ from **R-cat** to the category of functors $\mathbf{R} \longrightarrow \mathbf{Ab}$ as follows:

- $\mathbf{F}_M R = M$,
- $\mathbf{F}_M a = l_a$ for $a \in R$, where $l_a(m) = am$.

□

Hence, one can identify additive functors on $\mathbf{R} - \mathbf{cat}$ with the category of left R-modules.

3.2 Natural Transformations

Definition 3.13 If F and G are functors between categories $\mathbf{C}$ and $\mathbf{D}$ then a *natural transformation* $\eta : F \to G$ associates every object X in $\mathbf{C}$ a morphism $\eta_X : FX \to GX$ between objects of $\mathbf{D}$, such that for every morphism $f : X \to Y$ in $\mathbf{D}$ we have

$$\begin{array}{ccccc} FX & \xrightarrow{\eta_X} & GX & \qquad & X \\ \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle Gf} & & \downarrow{\scriptstyle f} \\ FY & \xrightarrow{\eta_Y} & GY & & Y \end{array}$$

is commutative. Furthermore, if η_X is an isomorphism for each object X, then we refer to it as a *natural equivalence.* $\diamondsuit$

Example 3.14 Let G be a group and define the category $\mathbf{G} - \mathbf{cat}$ as follows.

- $\star$ is the object
- G is the morphism

For the identity functor $\mathbf{1} : \mathbf{G} - \mathbf{cat} \longrightarrow \mathbf{G} - \mathbf{cat}$, a natural transformation η is an element of G such that for all $h \in G$

$$\begin{array}{ccc} \star & \xrightarrow{g} & \star \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ \star & \xrightarrow{g} & \star \end{array}$$

In other words, $hg = gh$, so $Hom_G(\mathbf{1}, \mathbf{1})$ is the center of G.

Example 3.15 Using the functors $F = Hom(N, -)$ and $G = Hom(M, -)$ for modules M and N and a module homomorphism $\phi : N \longrightarrow M$ we can define $\eta_X : Hom(M, X) \longrightarrow Hom(N, X)$ as $\eta_X \psi = \psi\phi$. For this to be a natural transformation we need for any $\lambda : X \to Y$, the following square to commute:

$$\begin{array}{ccc} Hom(M,X) & \xrightarrow{F\lambda} & Hom(M,Y) \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} \\ Hom(N,X) & \xrightarrow[G\lambda]{} & Hom(N,Y) \end{array}$$

Given $\psi \in Hom(N,X)$ we obtain the following diagram.

$$\begin{array}{ccc} \psi & \longrightarrow & \lambda\psi \\ \downarrow & & \downarrow \\ \psi\phi & \longrightarrow & \lambda(\psi\phi) = (\lambda\psi)\phi \end{array}$$

Thus the square commutes giving a natural transformation. □

Example 3.16 Natural transformation in **Quiv**(G)

Given two representations of a quiver of G, R and S, a natural transformation $\eta : R \longrightarrow S$ consists of linear transformations n_i (i a vertex of G) such that the following diagram commutes, for all i, j.

$$\begin{array}{c} i \\ \downarrow{\scriptstyle \alpha} \\ j \end{array} \qquad \begin{array}{ccc} R(i) & \xrightarrow{n_i} & S(i) \\ \downarrow{\scriptstyle R(\alpha)} & & \downarrow{\scriptstyle S(\alpha)} \\ R(j) & \xrightarrow{n_j} & S(j) \end{array}$$

• G with one edge $\quad \bullet_1 \xrightarrow{\alpha} \bullet_2$

The representation R would look like

$$V_1 \xrightarrow{R} V2$$

(Here R will denote both the representation and the linear transformation.) Likewise S would be

$$W_1 \xrightarrow{S} W_2$$

A natural transformation $\eta : R \longrightarrow S$ would then consist of an ordered pair of linear transformations A and B such that $BR = SA$ of the following diagram commutes.

$$\begin{array}{ccc} V_1 & \xrightarrow{R} & V_2 \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle B} \\ W_1 & \xrightarrow[S]{} & W_2 \end{array}$$

• G with two edges $\bullet_1 \xrightarrow{\alpha_1} \bullet_2 \xrightarrow{\alpha_2} \bullet_3$

The representation R would be

$$V_1 \xrightarrow{R_1} V_2 \xrightarrow{R_2} V_3$$

and the representation S would be

$$W_1 \xrightarrow{S_1} W_2 \xrightarrow{S_2} W_3$$

Here $\eta : R \longrightarrow S$ would be an ordered triple (n_1, n_2, n_3) of linear transformations such that

$$\begin{array}{ccccc} V_1 & \xrightarrow{R_1} & V_2 & \xrightarrow{R_2} & V_3 \\ \downarrow{\scriptstyle n_1} & & \downarrow{\scriptstyle n_2} & & \downarrow{\scriptstyle n_3} \\ W_1 & \xrightarrow{S_1} & W_2 & \xrightarrow{S_2} & W_3 \end{array}$$

commutes. We can then form a new category, called the category of F-representations of G. Here the objects are the functors $G \longrightarrow VecF$ and the morphisms are the natural transformations. For each functor the identity natural transformation makes sense, and composition of natural transformations results by composing the linear transformations at each vertex. □

Example 3.17 Given two functors F_i :**R-cat** $\longrightarrow$ **Ab**, $(i = 1, 2)$ we want to describe a natural transformation η from F_1 to F_2. Let $M_1 = F_1R$ and

$M_2 = F_2R$. For each $a \in R$ we want

$$\begin{array}{ccc} M_1 & \xrightarrow{n} & M_2 \\ {\scriptstyle F_1a}\downarrow & & \downarrow{\scriptstyle F_2a} \\ M_1 & \xrightarrow[n]{} & M_2 \end{array}$$

to commute. That would imply that n is a morphism in **Ab** and thus additive, and in turn would give us the following diagram.

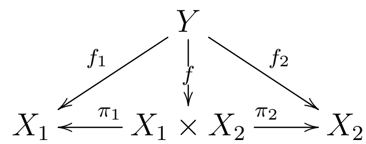
$$\begin{array}{ccc} x & \longrightarrow & n(x) \\ \downarrow & & \downarrow \\ ax & \longrightarrow & y \end{array}$$

Using the top side first we see that $y = an(x)$ and from the left side $y = n(ax)$. So $an(x) = n(ax)$ for all $a \in R$ and all $x \in M_1$. Implying that the natural transformation is just an R-module homomorphism. Hence, from 3.12, the category of all functors **R-cat**$\longrightarrow$**Ab** (where the objects are functors and the morphisms are the natural transformations) is equivalent to the category of left R-modules. Denoted as **Func**(**R-cat, Ab**) $\cong R-$ **Mod**.

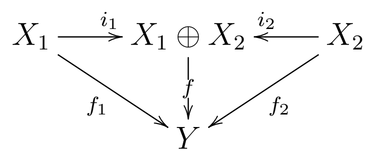
□

3.3 Products and Coproducts

Consider the following *dual* notions. Let **C** be a category with some objects X_1 and X_2. An object X is the *product* of X_1 and X_2 denoted as $X_1 \times X_2$ provided it satisfies the following universal property: there exists morphisms π_1 and π_2, called the canonical projection morphisms, such that for every object Y and pair of morphisms $f_1 : Y \to X_1$ and $f_2 : Y \to X_2$ there exists an unique morphism $f : Y \to X$ such that the following diagram commutes.



Similarly, an object X is the coproduct of objects X_1 and X_2 denoted as $X_1 \oplus X_2$, provided it satisfies the following universal property: there exists morphisms i_1 and i_2 such that for every object Y with $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ there exists a unique morphism $f : X_1 \oplus X_2 :\to Y$ the following diagram commutes.



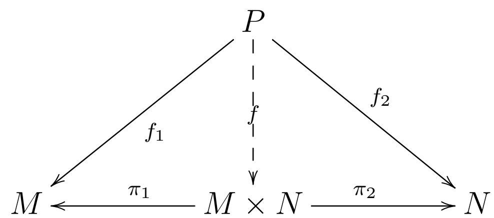
Example 3.18 Products and coproducts in **Set**

For a family of sets X_i the product is just simply the cartesian product of the sets. Resulting in $Y = \Pi_{i \in I} X_i = \{(x_i)_{i \in I} | x_i \in X_i \forall i\}$ with each $\pi_j : \Pi_{i \in I} X_i \longrightarrow X_j$ defined by $\pi_j((x_i)_{i \in I}) = x_j$. Given the set Y with any family of functions f_i for $i \in I$, the universal arrow $f : Y \longrightarrow \Pi_{i \in I} X_i$ is defined as $f(y) = (f_i(x_i))_{i \in I}$.

The coproduct of a family of sets X_i in **Set** is just simply the disjoint union of the sets with each i_j the inclusion map.

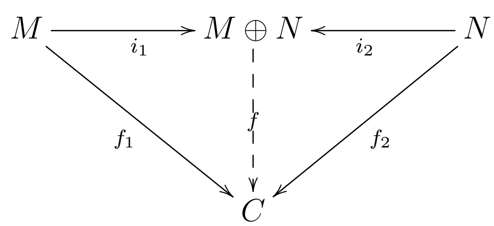
Example 3.19 Products and coproducts in $R-$**Mod**

In the category of $R-$**Mod**, we construct the product as the cartesian product of two modules. For given modules M and N, the product is, as sets, the ordinary cartesian product $M \times N$ where



commutes with $\pi_1 : M \times N \to M$ is defined as $\pi_1(x_1, x_2) = x_1$ and $\pi_2 : M \times N \to N$ is defined as $\pi_2(x_1, x_2) = x_2$ and $f(x) = (f_1(x), f_2(x))$.

The coproduct is constructed using the external direct sum of modules, which as sets is again the cartesian product. Given modules M and N, the coproduct is as a set is again the ordinary cartesian product but denoted $M \oplus N$ where



commutes with $i_1 : M \to M \oplus N$ defined as $i_1(x) = (x, 0)$ and $i_2 : N \to M \oplus N$ defined as $i_2(y) = (0, y)$ and $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$. $\square$

3.4 Kernels and Cokernels

Before we can discuss the universal property of kernel and cokernel, we will need to define a zero morphism.

Definition 3.20 Suppose **C** is a category, and $f : X \longrightarrow Y$ is morphism in **C**. The morphism f is called a *constant morphism* if for any object W in **C** and any morphisms $g, h \in Hom_{\mathbf{C}}(W, X)$, $fg = fh$. Dually, f is called a *coconstant morphism* is for any object Z in **C** and any $g, h \in Hom_{\mathbf{C}}(Y, Z)$, $gf = hf$. A *zero morphism* refers to a morphism which is both constant and coconstant. $\diamondsuit$

Definition 3.21 Suppose **C** is a category. An object I is called an *initial object* if for every object X in **C** there exits exactly one morphism $I \to X$. Dually, a *terminal object*, T, has precisely one morphism $X \to T$ for every object X. If an object is both terminal and initial is then called a *zero object*. $\diamondsuit$

If **C** has a zero object $\mathbf{0}$, given two objects X and Y in **C**, there are canonical morphisms $f : \mathbf{0} \longrightarrow X$ and $g : Y \longrightarrow \mathbf{0}$. Then fg is a zero morphism in $Hom_{\mathbf{C}}(Y, X)$.

A category with zero morphisms is one where, for any two objects A and B in **C**, there is a fixed morphism $0_{AB} : A \longrightarrow B$ such that for all objects X, Y, Z in **C** and all morphisms $f : Y \longrightarrow Z$ and $g : X \longrightarrow Y$, the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{0_{XY}} & Y \\ {\scriptstyle g}\downarrow & \searrow^{0_{XZ}} & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[0_{YZ}]{} & Z \end{array}$$

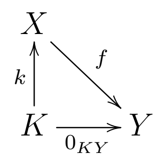
The morphisms 0_{XY} are forced to be zero morphisms and form a compatible system of such. If **C** is a category with zero morphisms, then the collection of 0_{XY} is unique. A category with a zero object is a category with zero morphisms (given by the compositions $0_{XY} : X \longrightarrow \mathbf{0} \longrightarrow Y$).

Example 3.22 In the category $R-$**Mod** a zero morphism is a module homomorphism $f : G \longrightarrow H$ that maps all of G to the identity element in H. The zero object in modules is the trivial module. $\square$

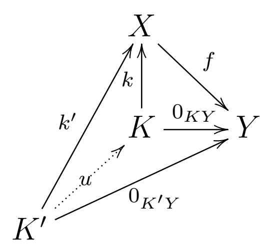
Example 3.23 The zero object in **Rep**(G) would be the zero vector space (the vector space only containing the zero vector) at each vertex and the zero transformation (the linear transformation that sends everything to the zero vector) for each edge. ∎

Definition 3.24 Let **C** be a category with zero morphisms with the following universal property: A *kernel of a morphism f* is any morphism $k : K \longrightarrow X$ such that:

- $f \circ k$ is the zero morphism from K to Y



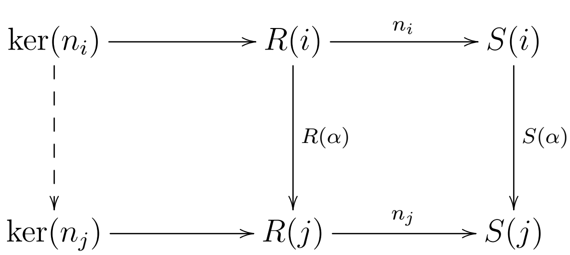
- Given any morphism $k' : K' \longrightarrow X$ such that $f \circ k'$ is the zero morphism, there is a unique morphism $u : K' \longrightarrow K$ such that $k \circ u = k'$. Resulting in the following commutative diagram.



◊

Example 3.25 ker *in* **Rep** (G)

The kernel in **Rep** (G) is computed at each vertex i. So given the commutative diagram

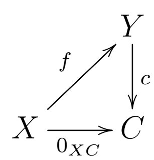


where $\ker(n_i)$ is the "standard" kernel of a linear transformation. This would then imply that there exists a unique linear transformation $\ker(n_i) \longrightarrow \ker(n_j)$ defined by the restriction of $R(\alpha)$ on $\ker(n_i)$ denoted as $\ker\alpha$. Thus $\ker\alpha$ is an object of **Rep** (G). □

The dual notion of *cokernel* is as follows:

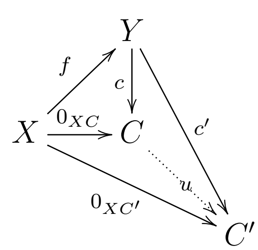
Definition 3.26 Let **C** be a category with zero morphisms and the following universal property: that the *cokernel* of a morphism $f : X \longrightarrow Y$ is an object C together with a morphism $c : Y \longrightarrow C$ such that

- the following diagram



  commutes.

- Moreover, any other such $c' : Y \longrightarrow C'$ with $0_{XC'} : X \to C'$ can be obtained by composing c with a unique morphism $u : C \longrightarrow C'$:

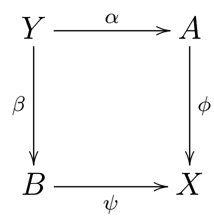


♢

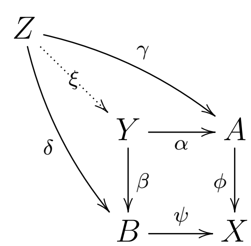
3.5 Pull-Backs and Push-outs

Here I will restrict my discussion of Category Theory to the category of R-**mod** since the development of the the functor *Ext* will be done with modules.

Definition 3.27 Given module homomorphisms $\phi : A \to X$ and $\psi : B \to X$, a *pull-back* of (ϕ, ψ) is an R-module Y with a pair of module homomorphisms $\alpha : Y \to A$ and $\beta : Y \to B$ such that $\phi\alpha = \psi\beta$ giving the following commutative diagram:

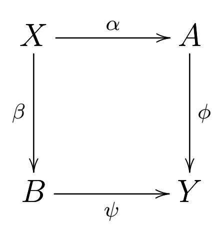


and has the following universal property: given $\gamma : Z \to A$ and $\delta : Z \to B$ with $\phi\gamma = \psi\delta$, there exists a unique module homomorphism $\xi : Z \to Y$ with $\delta = \beta\xi$ and $\gamma = \alpha\xi$ such that



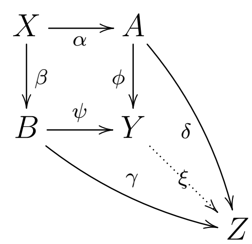
commutes.

Definition 3.28 Given module homomorphisms $\phi : X \to A$ and $\psi : X \to B$ a *push-out* of (ϕ, ψ) is a module Y with a pair of module homomorphisms $\alpha : A \to Y$ and $\beta : B \to Y$ such that $\phi\alpha = \psi\beta$ giving the following commutative diagram:



and has the following universal property: given $\gamma : B \to Z$ and $\delta : A \to Z$ with $\delta\alpha = \gamma\beta$, there exists a unique module homomorphism $\xi : Y \to Z$ with

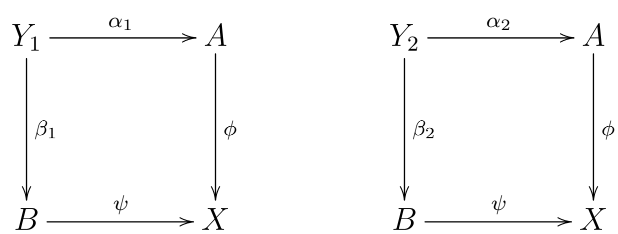
$\delta = \xi\phi$ and $\gamma = \psi\xi$ such that



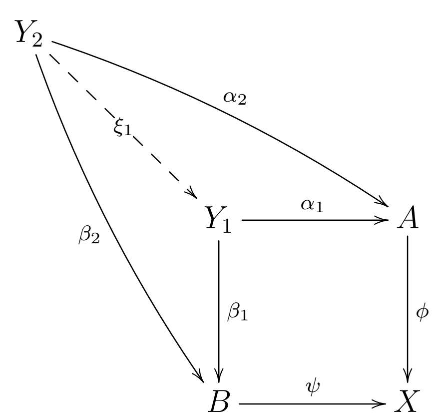
commutes.

Theorem 3.29 Pull-backs (or push-outs) are unique up to isomorphism.

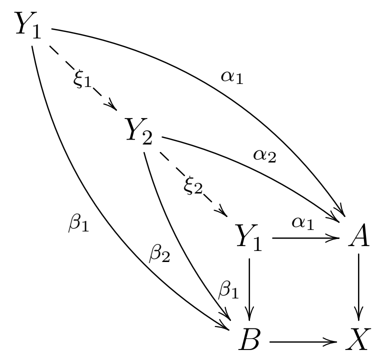
Proof: Given two pull-backs Y_1 and Y_2, we have the following commutative squares.



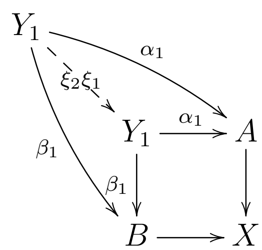
We can then create the following commutative diagram



where $\xi_1 : Y_2 \to Y_1$ is unique. We can also create another diagram with Y_1 and Y_2 reversed, resulting in another unique morphism ξ_2. Now consider the following commutative diagram.



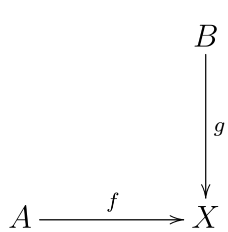
Considering the outer path we have



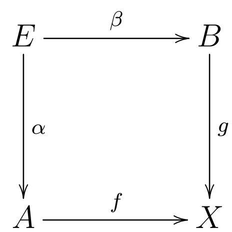
Since $\alpha_1 1_A = \alpha_1$ and $\alpha_1\xi_2\xi_1 = \alpha_1$ uniqueness would imply that $\xi_2\xi_1 = 1_{Y_1}$. By symmetry we find that $\xi_1\xi_2 = 1_{Y_2}$. Thus from the fact that these morphisms "undo" each other and they are unique, we see that ξ_1 and ξ_2 are isomorphisms and hence $Y_1 \cong Y_2$. (An analogous argument can be made for push-outs.) Therefore, we can refer to *the* pull-back (or push-out). ∎

Example 3.30 Pull-back and Push-out in $R - \mathbf{Mod}$

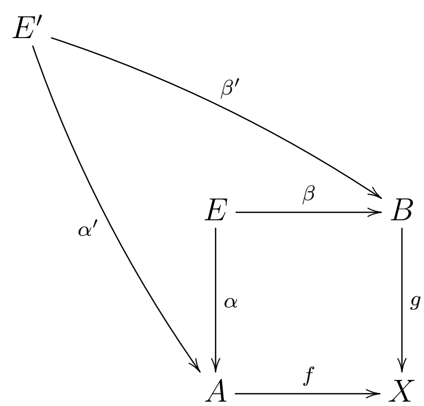
Given



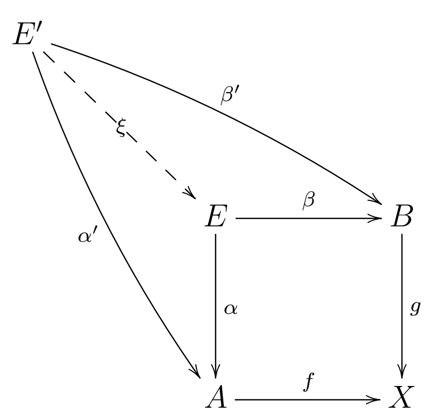
For $A \oplus B$, define a subset $E = \{(a, b)|f(a) = g(b)\}$. Here E is the kernel of $A \oplus B \xrightarrow{[f,-g]} X$, so we get that E is a submodule of $A \oplus B$. Now define maps $\alpha : E \longrightarrow A$ and $\beta : E \longrightarrow B$ by $\alpha(a,b) = a$ and $\beta(a,b) = b$ respectively. Clearly, these would be module homomorphisms. For some $(a,b) \in E$ we have $f\alpha(a,b) = fa$ and $g\beta(a,y) = gb$. However, by definition of E, $fa = gb$, so the following square commutes.



Suppose there exists $\alpha' : E' \to A$ and $\beta' : E' \to B$ giving the following commutative diagram.



We need to construct a unique $\xi : E' \to E$ such that



commutes. For $\xi(e') = (a, b) \in E$, we would need

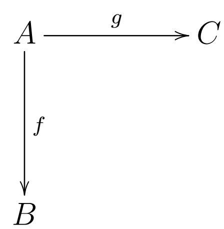
$$\begin{aligned} \beta'(e') &= \beta\xi(e') \\ &= \beta(a, b) \\ &= b \end{aligned}$$

and similarly, $\alpha'(e') = a$. Resulting in ξ being defined as

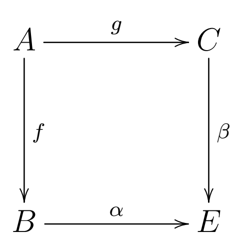
$$\xi(e') = (\alpha'(e'), \beta(e')).$$

Since $f\alpha' = g\beta'$ we also get $\xi(e') \in E$. Thus ξ is the unique morphism making the diagram commute.

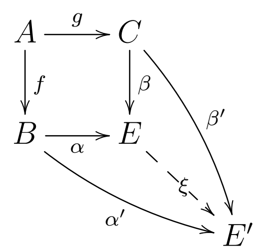
Now given



Let $S = \{(f(a), -g(a)) \in B \oplus C | a \in A\}$. Since addition is defined componentwise and f and g are module homomorphisms, we can see that S is a submodule of $B \oplus C$. Now define $E = (B \oplus C)/S$ (giving E as the cokernel of $A \xrightarrow{[f,-g]^T} A \oplus B$), and maps $\alpha : B \to E$ and $\beta : C \to E$ by $b \mapsto (b, 0) + S$ and $c \mapsto (0, c) + S$ for $b \in B$ and $c \in C$. One can see these are then module homomorphisms. Now given $a \in A$, $f(a) \mapsto (f(a), 0)$ by α. Since $(f(a), -g(a)) \in S$, we get $(f(a), 0) + S = (0, g(a)) + S$. Which implies that $\alpha f(a) = \beta g(a)$. Thus $\beta g = \alpha f$, and we have create the following commutative square.



Suppose there exists α' and β' such that $\beta' g = \alpha' f$. We again need to create a ξ such that the following diagram commutes.



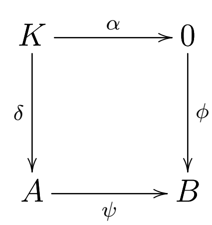
Then for $b \in B$ we have

$$\begin{aligned} \alpha'(b) &= \xi\alpha(b) \\ &= \xi((b, 0) + S) \end{aligned}$$

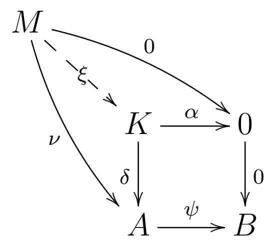
and for $c \in C$, we get $\beta'(c) = \xi((0, c)+S)$. This would imply that $\xi((b, c)+S) = \xi((b, 0) + S, (0, c) + S) = \alpha'(b) + \beta'(c)$. Thus ξ is uniquely defined, giving the desired commutativity.

Notice that the pull-back requires kernels and products and the push-out requires cokernels and coproducts.

Now consider the following pull-back diagram of modules:

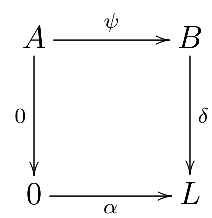


Given $M \xrightarrow{\nu} A \xrightarrow{\psi} B$, where $\psi\nu = 0$, we get the following commutative diagram.

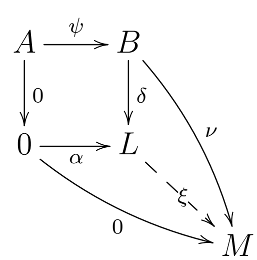


Hence we get a unique homomorphism $\xi : M \to K$ such that $\nu = \delta\xi$ and $\alpha\xi = 0$, which satisfies the universal property of the kernel. Thus $K \to A$ is the kernel of $A \to B$.

For the dual notion, consider the following push-out of modules.

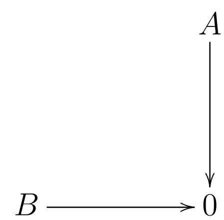


Now given $A \xrightarrow{\psi} B \xrightarrow{\nu} M$ where $\nu\psi = 0$, we get the following commutative diagram.

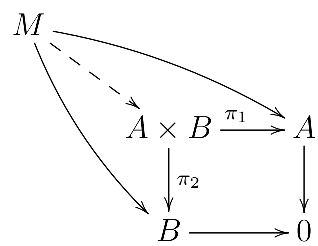


Hence we get a a unique homomorphism $\xi : L \to M$ such that $\nu = \xi\delta$ and $0 = \xi\alpha$. Which satisfies the universal property of cokernel. Thus $B \to L$ is the cokernel of $A \to B$.

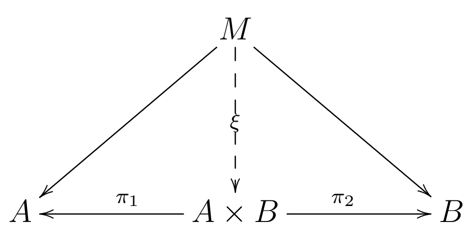
Now consider



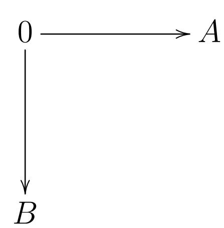
Here the pull-back is simply the product $A \times B$. Using $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B :\to B$ as the projection maps, the universal property of pull-back says there exists a unique $\xi : M \to A \times B$ such that



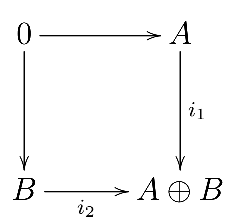
commutes. Which gives the following commutative diagram.



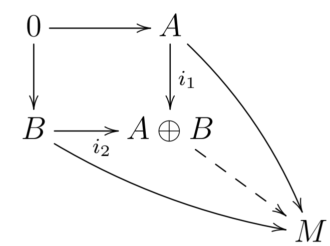
Thus satisfying the universal property of the product. Dually, the push-out of



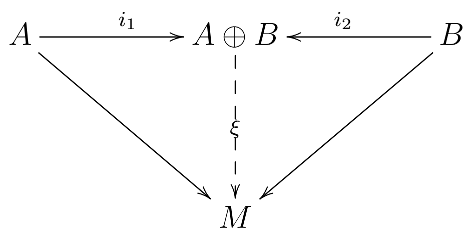
using the injections maps will yield the coproduct (direct sum)



By the universal property of push-out, we get a unique $\xi : A \oplus B \to M$ such that



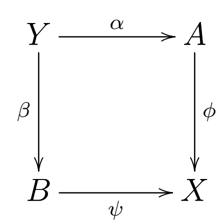
commutes. Which give the following commutative diagram.



Thus satisfying the universal property of the coproduct.

At this point it is natural to ask how these ideas will apply to the notion of sequences of modules. We now present the following theorem.

Theorem 3.31 The square

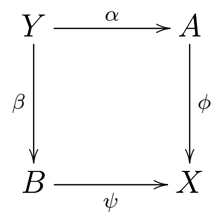


is a pull-back diagram if and only if the sequence

$$0 \longrightarrow Y \xrightarrow{\{\alpha,\beta\}} A \oplus B \xrightarrow{<\phi,-\psi>} X$$

is exact. Where $\{\alpha, \beta\}$ is defined as $\{\alpha, \beta\}(a) = (\alpha(a), \beta(a))$ and $< \phi, -\psi >$ is defined as $< \phi, -\psi > (a, b) = \phi(a) - \psi(b)$.

Proof: Assuming

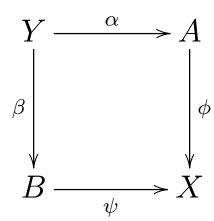


is a pull-back diagram we have, $Y = \ker < \phi, -\psi >$ from Example 3.30. Thus $\operatorname{Im}\{\alpha, \beta\} = \ker < \phi, -\psi >$.

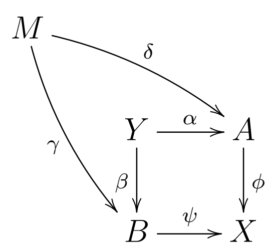
Now assuming

$$0 \longrightarrow Y \xrightarrow{\{\alpha,\beta\}} A \oplus B \xrightarrow{<\phi,-\psi>} X$$

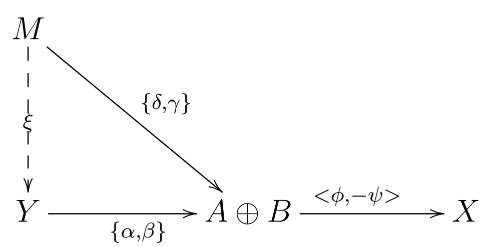
is exact, we have $< \phi, -\psi > \circ\{\alpha, \beta\} = 0$. Giving the commutative square.



Now suppose there exists a module, M, such that



commutes. That would imply $< \phi, -\psi > \circ\{\delta, \gamma\} = 0$. By the universal property of the kernel there exists a unique ξ induced by $Y \to A \oplus B$ giving



It then follows that $\{\alpha, \beta\} \circ \xi = \{\delta, \gamma\}$ and thus

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & A \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \phi} \\ B & \xrightarrow[\psi]{} & X \end{array}$$

is a pull-back diagram. ∎

The dual notion is as follows.

Theorem 3.32 The square

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & A \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \phi} \\ B & \xrightarrow[\psi]{} & Y \end{array}$$

is a push-out diagram if and only if the sequence

$$X \xrightarrow{\{\alpha,\beta\}} A \oplus B \xrightarrow{<\phi,-\psi>} Y \longrightarrow 0$$

is exact.

Proof: Assume that

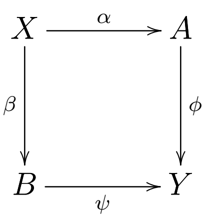
$$\begin{array}{ccc} X & \xrightarrow{\alpha} & A \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \phi} \\ B & \xrightarrow[\psi]{} & Y \end{array}$$

is a push-out diagram. Then from Example 3.30, we can see that $Y = \operatorname{cok}\{\alpha, \beta\}$ with the corresponding ϕ and ψ and thus
$\operatorname{Im}\{\alpha, \beta\} = \ker < \phi, -\psi >$.

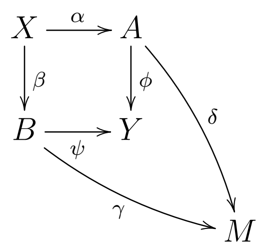
Now assuming that

$$toX \xrightarrow{\{\alpha,\beta\}} A \oplus B \xrightarrow{<\phi,-\psi>} Y \longrightarrow 0$$

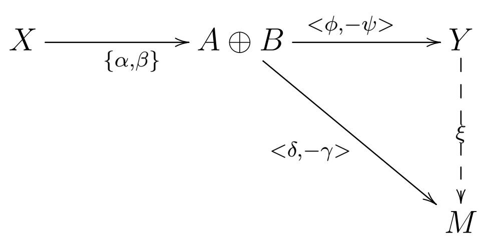
is exact, we get $< \phi, -\psi > \circ \{\alpha, \beta\} = 0$. Thus the following diagram commutes.



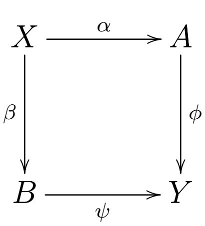
Now suppose there exists a module, M, such that



commutes. That would imply $< \delta, -\gamma > \circ \{\alpha, \beta\} = 0$. By the universal property of the cokernel, there exists a unique ξ induced by $A \oplus B \to Y$, giving the following diagram.



It then follows that $< \phi, -\psi > \circ \xi = < \delta, -\gamma >$ and thus



is a push-out diagram. ∎

Chapter 4

Extensions of Modules

Through out the field of algebra it is natural to look at subgroups, subrings, and submodules and ask questions about the associated quotients. In homology we narrow our focus down to just modules. Now given two modules, A, and B over a fixed ring R, our goal is to find all modules E such that B is a submodule of E and $E/B \cong A$.

4.1 Extensions

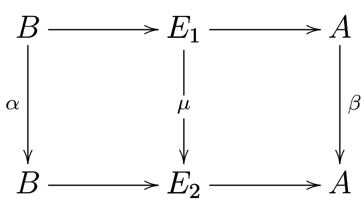
More generally, given R-modules A and B and a short exact sequence

$$0 \longrightarrow B \xrightarrow{\psi} E \xrightarrow{\phi} A \longrightarrow 0$$

Then by the First Isomorphism Theorem, $E/\ker\phi \cong A$ gives $E/\operatorname{Im}\psi \cong A$ and also $B \cong \operatorname{Im}\psi$ since ψ is one-to-one. Such a sequence is called an extension of B by A.

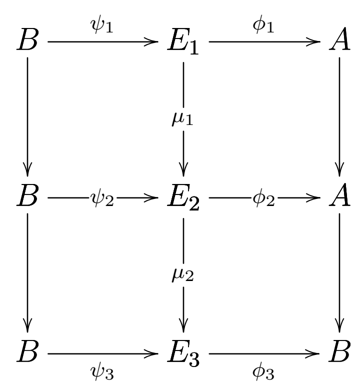
There is an equivalence relation on such sequences. Define the relation

$\backsim$ such that $E_1 \backsim E_2$ where

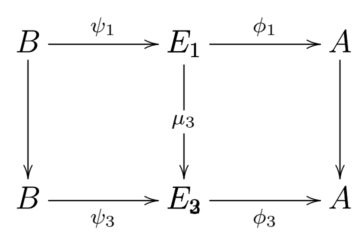


is commutative where α and β are isomorphisms.

To show that $\backsim$ is reflexive, let α, β, and μ be the identity functions. Such a diagram is obviously commutative. For transitivity, let $E_1 \backsim E_2$ and $E_2 \backsim E_3$, so there exists μ_1 and μ_2 such that the following diagram is commutative.



Define the homomorphism $\mu_3 : E_1 \longrightarrow E_3$ to be $\mu_3 = \mu_1 \circ \mu_2$, since composition of two homomorphism is a homomorphism this composition makes sense. This would give us the following diagram.



Due to the commutativity of the previous diagrams, it follows that this diagram would be also commutative. Thus if $E_1 \backsim E_2$ and $E_2 \backsim E_3$, then $E_1 \backsim E_3$. Finally, if $E_1 \backsim E_2$, then it is necessary by Lemma 2.19 that μ is also an isomorphism. Thus using α^{-1}, β^{-1}, and μ^{-1} makes $\backsim$ symmetric and hence an equivalence relation.

Let $E(A, B)$ denote the equivalence class of all such extension of A by B. This will always contain at least one element, namely the direct sum of modules. Given A and B, R-modules we can create the sequence

$$A \xrightarrow{i_A} A \oplus B \xrightarrow{\pi_B} B$$

where if $a \in A$ and $b \in B$ then $(a, b) \in A \oplus B$. Here i_A would be the canonical injection map from A to $A \oplus B$ and π_B would be the canonical projection map from $A \oplus B$ to B. i_A would be clearly one-to-one and π_B is clearly onto. Since for each $a \in A$, $i_A(a) = (a, 0)$ and for each $b \in B$, $\pi_B(a, b) = b$ we get $\operatorname{Im} i_A = \ker \pi_B$. In addition, using π_A as the projection map on A, we get $\pi_A i_A = 1_A$ where 1_A is the identity on A. Similarly, $\pi_B i_B = 1_B$. Thus we get a splitting sequence. Such an extension is called a *split extension.*

Example 4.1 $E(\mathbb{Z}_2, \mathbb{Z}_4)$

As $\mathbb{Z}$-modules, we are looking for all sequences of the form

$$0 \longrightarrow \mathbb{Z}_4 \xrightarrow{\alpha} E \xrightarrow{\beta} \mathbb{Z}_2 \longrightarrow 0.$$

Where, by the first isomorphism theorem, $E/\ker\beta \cong \mathbb{Z}_2$. Since $\ker\beta = \operatorname{Im}\alpha \cong \mathbb{Z}_4$, we have $E/\mathbb{Z}_4 \cong \mathbb{Z}_2$. Thus $|E|/4 = 2$, which implies that $|E| = 8$.

The only abelian groups of order 8 are $\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. For $\mathbb{Z}_4 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ there exist no one-to-one maps, since $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ has no order 4 elements.

As we just saw $E(\mathbb{Z}_2, \mathbb{Z}_4)$ contains the split extension

$$0 \longrightarrow \mathbb{Z}_4 \xrightarrow{i} \mathbb{Z}_4 \oplus \mathbb{Z}_2 \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0.$$

As for another possibility, we can use

$$0 \longrightarrow \mathbb{Z}_4 \xrightarrow{\nu} \mathbb{Z}_8 \xrightarrow{\pi} \mathbb{Z}_2 \longrightarrow 0,$$

where $\nu(j) = 2j$. Here ν is clearly one-to-one with

$$\begin{aligned} \operatorname{Im}\nu &= \{0, 2, 4, 6\} \\ &= \ker\pi \end{aligned}$$

For any other possibilities with $E \cong \mathbb{Z}_8$ consider $Hom(\mathbb{Z}_4, \mathbb{Z}_8)$. We know that $Hom(\mathbb{Z}_4, \mathbb{Z}_8) \cong \mathbb{Z}_4$ and thus $Hom(\mathbb{Z}_4, \mathbb{Z}_8)$ has only 4 elements. These can be determined by where each homomorphism sends 1. Resulting in:

(i) $1 \to 2$

(ii) $1 \to 4$

(iii) $1 \to 6$

(iv) $1 \to 0$

(*iv*) is clearly not one-to-one, and (*ii*) gives $2 \mapsto 0$, so it is not one-to-one. Now we will need to match the remaining maps to corresponding elements of $Hom(\mathbb{Z}_8, \mathbb{Z}_2)$ to create a short exact sequence. The elements of $Hom(\mathbb{Z}_8, \mathbb{Z}_2)$ are:

(a) $k \to k$

(b) $k \to 0$

(b) is clearly not onto, so we only need to consider (a). Now $a \mapsto 2a$ and

$a \mapsto 6a$ will have the same image since $6 = -2$. Giving the following diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}_4 & \xrightarrow{\nu} & \mathbb{Z}_8 & \xrightarrow{\pi} & \mathbb{Z}_2 \\ & & \downarrow{\scriptstyle \alpha} & & \| & & \| \\ 0 & \longrightarrow & \mathbb{Z}_4 & \xrightarrow{\gamma} & \mathbb{Z}_8 & \xrightarrow{\pi} & \mathbb{Z}_2 \end{array}$$

Where γ is defined as $a \mapsto -2a$ and α as $a \mapsto -a$. By Theorem 2.19, we get α as an isomorphism giving us the same sequence. Therefore, up to isomorphism, there is only one extension with middle term $\mathbb{Z}_8$.

For any other possibilities with $E \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$, consider $Hom(\mathbb{Z}_4, \mathbb{Z}_4 \oplus \mathbb{Z}_2) \cong Hom(\mathbb{Z}_4, \mathbb{Z}_4) \oplus Hom(\mathbb{Z}_4, \mathbb{Z}_2) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$.

(i) $1 \rightarrow (1, 0)$

(ii) $1 \rightarrow (1, 1)$

(iii) $1 \rightarrow (2, 0)$

(iv) $1 \rightarrow (2, 1)$

(v) $1 \rightarrow (3, 0)$

(vi) $1 \rightarrow (3, 1)$

(vii) $1 \rightarrow (0, 0)$

(viii) $1 \rightarrow (0, 0)$

(vii) and $(viii)$ are clearly not one-to-one and (iii) and (iv) are not one-to-one since $2 \mapsto (0, 0)$. Thus we need to match $(i), (ii), (v)$, and (vi) with homomorphisms from $Hom(\mathbb{Z}_4 \oplus \mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. These homomorphism are:

(a) $(1,0) \to 1$

$(0,1) \to 1$

(b) $(1,0) \to 1$

$(0,1) \to 0$

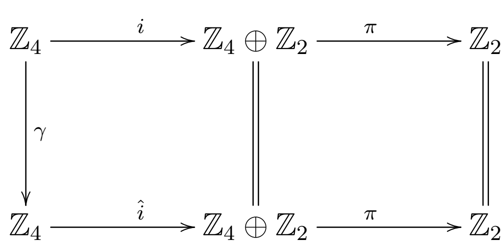
(c) $(1,0) \to 0$

$(0,1) \to 1$

(d) $(1,0) \to 0$

$(0,1) \to 0$

Here (a) is clearly not onto so we can ignore it. For (b) the kernel is $< 2 > \oplus \mathbb{Z}_2$ which is $\{(0,0),(2,0),(0,1),(2,1)\}$. This has no match from $Hom(\mathbb{Z}_4, \mathbb{Z}_4 \oplus \mathbb{Z}_2)$, so we can disregard it. Now for (c) we get the kernel as $\mathbb{Z}_4 \oplus 0$. This will match with homomorphisms (i) and (v) from $Hom(\mathbb{Z}_4, \mathbb{Z}_4 \oplus \mathbb{Z}_2)$. Since $1 \mapsto (3,0) = (-1,0)$. With these can create the following commutative diagram.



Where $\hat{i}$ is defined as $\hat{i}(a) = (-a, 0)$ and $\gamma(a) = -a$. By Theorem 2.19 we get that γ is an isomorphism. Now for homomorphism (d) we get $(a,b) = a + b$, so the kernel is a cyclic group of order 4 generated by $(1,1)$. giving the kernel as $\{(0,0),(1,1),(2,0),(3,1)\}$. We can then match this with homomorphisms (ii) and (vi). With these sequences the following commutative diagram can be created.

$$\begin{array}{ccccc}
\mathbb{Z}_4 & \xrightarrow{\alpha} & \mathbb{Z}_4 \oplus \mathbb{Z}_2 & \xrightarrow{\beta} & \mathbb{Z}_2 \\
\downarrow{\scriptstyle \gamma} & & \| & & \| \\
\mathbb{Z}_4 & \xrightarrow{\hat{\alpha}} & \mathbb{Z}_4 \oplus \mathbb{Z}_2 & \xrightarrow{\beta} & \mathbb{Z}_2
\end{array}$$

Where γ is defined as $a \mapsto -a$, $\alpha(a) = (a, a)$, and $\hat{\alpha}(a) = (-a, a)$. With β defined as $(a, b) \mapsto a + b$. Again by Theorem 2.19 we get that γ is an isomorphism.

The question remains, is there an isomorphism between

$$\mathbb{Z}_4 \xrightarrow{i} \mathbb{Z}_4 \oplus \mathbb{Z}_2 \xrightarrow{\pi} \mathbb{Z}_2$$

and

$$\mathbb{Z}_4 \xrightarrow{\alpha} \mathbb{Z}_4 \oplus \mathbb{Z}_2 \xrightarrow{\beta} \mathbb{Z}_2,$$

with the previously defined α and β. We can then define a homomorphism $\xi : \mathbb{Z}_4 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \oplus \mathbb{Z}_2$ by $\xi(a, b) = (a, a+b)$, creating the following commutative diagram.

$$\begin{array}{ccccc}
\mathbb{Z}_4 & \xrightarrow{i} & \mathbb{Z}_4 \oplus \mathbb{Z}_2 & \xrightarrow{\pi} & \mathbb{Z}_2 \\
\| & & \downarrow{\scriptstyle \xi} & & \| \\
\mathbb{Z}_4 & \xrightarrow{\alpha} & \mathbb{Z}_4 \oplus \mathbb{Z}_2 & \xrightarrow{\beta} & \mathbb{Z}_2
\end{array}$$

Once again, by Theorem 2.19, we get ξ as an isomorphism. Thus for up to isomorphism, the only extension with middle term $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ is the split extension. Resulting in only two elements for $E(\mathbb{Z}_2, \mathbb{Z}_4)$. ∎

Our goal from here is to make $E(-, -)$ a bifunctor. To do this the following lemmas will come in useful.

Lemma 4.2 If the square

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & A \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\phi} \\ B & \xrightarrow[\psi]{} & X \end{array}$$

is a pull-back diagram, then

(i) β induces $ker\alpha \xrightarrow{\backsim} \ker\psi$;

(ii) if ψ is an epimorphism, then so is α.

Proof: Part (i): Let L be the kernel of α and K be the kernel of ψ. Giving the following diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \xrightarrow{\nu} & Y & \xrightarrow{\alpha} & A \\ & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\phi} \\ 0 & \longrightarrow & K & \xrightarrow{\lambda} & B & \xrightarrow{\psi} & X \end{array}$$

From the commutativity of the pull-back, we get

$$\begin{aligned} \psi\beta\nu &= \phi\alpha\nu \\ &= \phi 0 \\ &= 0 \end{aligned}$$

Thus by the universal property of the pull-back, there exists a unique δ such that $\lambda\delta = \beta\nu$. We saw in a previous chapter that up to isomorphism the pull-back of modules is $Y = \{(a, b) \in A \oplus B | \phi(a) = \psi(b)\}$ with $\alpha(a, b) = a$ and $\beta(a, b) = b$. Thus $(a, b) \in \ker\alpha$ if and only if $a = 0$. However, for $(a, b) \in Y$ we get $(0, b) \in Y$ and thus $\phi(0) = \psi(b)$. Which implies that $\psi(b) = 0$. Resulting in $\ker\alpha = \{(0,b) \in A \oplus B | b \in \ker\psi\}$. This would make δ to be β restricted to $\ker\alpha$, giving $\ker\alpha \cong \ker\psi$.

Note: The induced isomorphism on the kernels can be replaced by equality as follows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \xrightarrow{\nu\delta^{-1}} & Y & \xrightarrow{\alpha} & A \\ & & \| & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \phi} \\ 0 & \longrightarrow & K & \xrightarrow{\lambda} & B & \xrightarrow{\psi} & X \end{array}$$

Meaning, we can just assume that the kernels are equal when pull-backs are taken. Also, since $\nu\delta^{-1}$ is the kernel of α the sequence

$$0 \longrightarrow K \longrightarrow Y \longrightarrow A$$

is exact.

<u>Part (ii):</u> Consider the sequence $0 \longrightarrow Y \xrightarrow{\{\alpha,\beta\}} A \oplus B \xrightarrow{<\phi,-\psi>} X$ which is exact by Lemma 3.31. Suppose $a \in A$ and consider the following diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & A \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \phi} \\ B & \xrightarrow[\psi]{} & X \end{array}$$

ψ is epimorphic, which implies there exists a $b \in B$ such that $\phi(a) = \psi(b)$.

$$\phi(a) = \psi(b)$$

$$\Longrightarrow \phi(a) - \psi(b) = 0$$

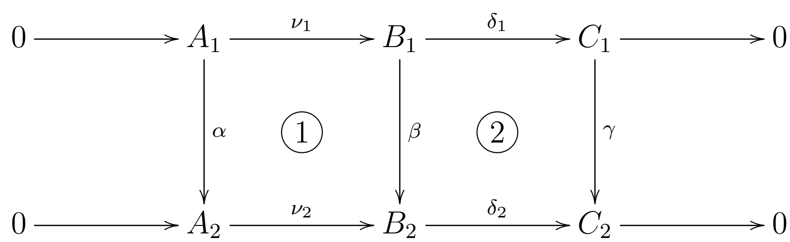
and thus $(a,b) \in \ker < \phi, -\psi >$ and from exactness of the sequence, $\ker < \phi, -\psi > = \mathrm{im}\{\alpha, \beta\}$, thus there exists $y \in Y$ with $a = \alpha(y)$ and hence α is epimorphic.

Note: In this case one obtains, from the pull-back, a short exact sequence

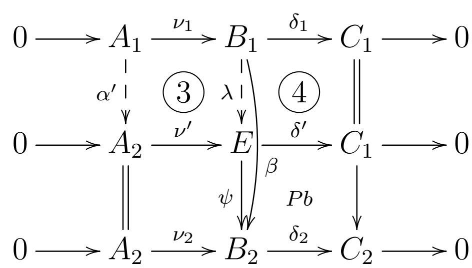
$$0 \longrightarrow K \longrightarrow Y \longrightarrow A \longrightarrow 0.$$

∎

Now given two exact sequences and connecting maps.

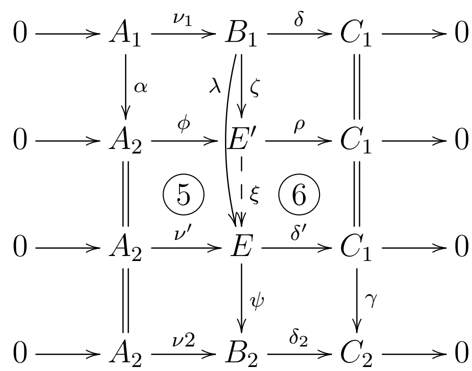


Let E be the pull-back of δ_2 and γ, so we get a δ and ψ with the following diagram.

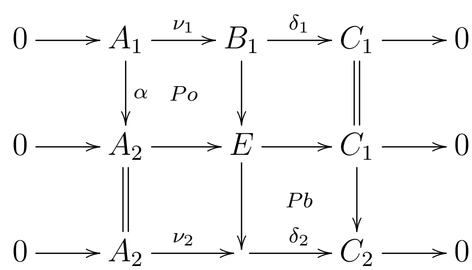


By the commutativity of (1), we are given an induced map into E from the universal property of the pull-back, such that $\lambda\delta' = \delta$ and $\psi\lambda = \beta$. Thus (4) commutes and $\delta\lambda\nu_2 = \delta_2 = \nu_2 = 0$. Resulting in a unique morphism α' into A_2, the kernel of δ', with $\lambda\nu_2 = \nu'\alpha'$, making (3) commute. Now $\nu_1\alpha' = \psi\nu'\alpha' = \psi\lambda\nu_2 = \beta\nu_2 = \nu_1\alpha$. Since ν_1 is a monomorphism, we can cancel, thus $\alpha = \alpha'$. By the ending note in Lemma 4.2, we get that the middle sequence is then exact.

Now let E' be the push-out of ν_1 and α creating the following diagram.

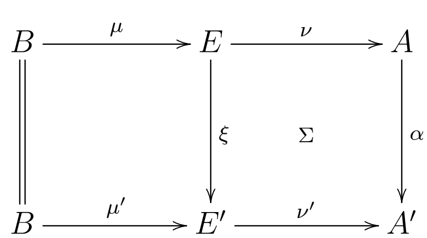


Since $\nu' = \xi\phi$ by the universal property of push-out, we know that square (5) will then commute. The universal property of the cokernel then gives us that $\rho = \delta'\xi^{-1}$ and thus square (6) commutes. This would also imply that the sequence $0 \longrightarrow A_2 \longrightarrow E' \longrightarrow C_2 \longrightarrow 0$ is also exact. Thus by Lemma 2.19, we get that ξ is an isomorphism. Giving the following commutative diagram with exact rows.



We will use this idea to show the following lemma.

Lemma 4.3 Let



be a commutative diagram with exact rows. Then the square Σ is both a pull-back and push-out diagram.

Proof: Let $\hat{E}$ be the pull-back of ν' and α. Then we get the following commutative diagram.

$$
\begin{array}{ccccc}
B & \longrightarrow & E & \longrightarrow & A \\
\| & & \downarrow & & \| \\
B & \longrightarrow & \hat{E} & \longrightarrow & A \\
\| & & \downarrow & Pb & \downarrow \alpha \\
B & \longrightarrow & E' & \xrightarrow{\nu'} & A'
\end{array}
$$

By Lemma 2.19 this forces $E \longrightarrow \hat{E}$ to be an isomorphism, thus Σ was originally the pull-back. Now by Theorem 3.31 there is an exact sequence

$$0 \longrightarrow E \longrightarrow A \oplus E' \longrightarrow A'.$$

However, ν' is onto and thus $A \oplus E' \longrightarrow A'$ is also onto, so by Theorem 3.32 we get Σ is also a push-out diagram.

∎

Lemma 4.4 Let

$$
\begin{array}{ccccc}
B & \xrightarrow{\mu} & E & \xrightarrow{\nu} & A \\
\downarrow \beta & \Sigma & \downarrow \xi & & \| \\
B' & \xrightarrow{\mu'} & E' & \xrightarrow{\nu'} & A
\end{array}
$$

be a commutative diagram with exact row. Then the square Σ is both a push-out and pull-back diagram.

Proof: Let $\hat{E}$ be the push-out of β and μ. This would result in the following commutative diagram.

$$\begin{array}{ccccc} B & \xrightarrow{\mu} & E & \longrightarrow & A \\ \downarrow{\scriptstyle \beta} & Po & \downarrow & & \| \\ B' & \longrightarrow & \hat{E} & \longrightarrow & A \\ \| & & \downarrow & & \| \\ B' & \longrightarrow & E' & \longrightarrow & A \end{array}$$

By Theorem 2.19 we get $\hat{E} \longrightarrow E$ as an isomorphism and thus Σ was originally the push-out. Now by Theorem 3.32 we get an exact sequence

$$B \longrightarrow B' \oplus E \longrightarrow E' \longrightarrow 0.$$

However, μ was one-to-one, implying that $B' \longrightarrow B \oplus E$ is also one-to-one. Thus by Theorem 3.31, Σ is also a pull-back diagram. ∎

Back to $E(-,-)$. Let $\alpha : A' \to A$ be a homomorphism and let $B \xrightarrow{\kappa} E \xrightarrow{\nu} A$ be a representative element in $E(A, B)$ and consider the diagram:

$$\begin{array}{ccccc} & & E^{\alpha} & \overset{\nu'}{\dashrightarrow} & A' \\ & & \vdots\,{\scriptstyle \xi} & & \downarrow{\scriptstyle \alpha} \\ B & \xrightarrow{\kappa} & E & \xrightarrow{\nu} & A \end{array}$$

Where $(E^{\alpha}; \nu', \xi)$ is the pull-back of (α, ν). By Lemma 4.2 we obtain the extension $B \longrightarrow E^{\alpha} \xrightarrow{\nu'} A'$. Allowing us to define:

$$\alpha^* : E(A, B) \to E(A', B)$$

by assigning $B \longrightarrow E^{\alpha} \longrightarrow A'$ to the class $B \longrightarrow E \longrightarrow A$. This definition is independent of our choice of representative since pull-backs of equivalent short exact sequences will result in equivalent pull-back sequences.

We claim that this definition of $\alpha^* = E(\alpha, B)$ makes $E(-, B)$ into a contravariant functor from modules to sets. First we check to see if the identity is preserved. So for $\alpha = 1_A : A \to A$ we have the following diagram:

$$\begin{array}{ccccc} & & E^{1_A} & \xrightarrow{\nu} & A \\ & & \| & & \| \\ B & \xrightarrow{\kappa} & E & \xrightarrow{\nu} & A \end{array}$$

Giving back the original sequence. Now we need to check that composition is preserved. So let $\alpha' : A'' \to A'$ and $\alpha : A' \to A$. Our goal is to show that $E(\alpha \circ \alpha', B) = E(\alpha', B) \circ E(\alpha, B)$ or that in the following diagram where each square is a pull-back

$$\begin{array}{ccc} (E^{\alpha})^{\alpha'} & \overset{\nu''}{\dashrightarrow} & A'' \\ {\scriptstyle \beta'} \downarrow & & \downarrow {\scriptstyle \alpha'} \\ E^{\alpha} & \overset{\nu'}{\dashrightarrow} & A' \\ {\scriptstyle \beta} \downarrow & & \downarrow {\scriptstyle \alpha} \\ E & \xrightarrow{\nu} & A \end{array}$$

the composite square

$$\begin{array}{ccc} (E^{\alpha})^{\alpha'} & \overset{\nu''}{\dashrightarrow} & A'' \\ \downarrow & & \downarrow {\scriptstyle \alpha \circ \alpha'} \\ E & \xrightarrow{\nu} & A \end{array}$$

is a pull-back of $(\nu, \alpha \circ \alpha')$. To do this, construct the following commutative diagram with exact rows.

$$\begin{array}{ccccc} B & \longrightarrow & (E^{\alpha})^{\alpha'} & \xrightarrow{\nu''} & A'' \\ \| & & \downarrow {\scriptstyle \beta'} & & \downarrow {\scriptstyle \alpha} \\ B & \longrightarrow & E^{\alpha} & \xrightarrow{\nu'} & A' \\ \| & & \downarrow {\scriptstyle \beta} & & \downarrow {\scriptstyle \alpha} \\ B & \longrightarrow & E & \xrightarrow{\nu} & A \end{array}$$

Giving the following diagram with exact rows.

$$
\begin{array}{ccccc}
B & \longrightarrow & (E^{\alpha})^{\alpha} & \longrightarrow & A'' \\
\| & & \downarrow & \Sigma & \downarrow \\
B & \longrightarrow & E & \longrightarrow & A
\end{array}
$$

Thus by Lemma 4.3 we get Σ as a pull-back. Resulting in $E(-,B)$ as a contravariant functor from the category modules to the category of sets.

Let $\beta : B \to B'$ be a homomorphism and let $B \xrightarrow{\kappa} E \xrightarrow{\nu} A$ be a representative of $E(A,B)$ and consider the diagram:

$$
\begin{array}{ccccc}
B & \xrightarrow{\kappa} & E & \xrightarrow{\nu} & A \\
\downarrow{\scriptstyle \beta} & & \vdots{\scriptstyle \xi} & & \\
B' & \xrightarrow{\kappa'} & E_{\beta} & &
\end{array}
$$

where $(E_\beta; \kappa', \xi)$ is the push-out of (β, κ). The dual of Lemma 4.2 shows that we obtain an extension $B' \longrightarrow E_\beta \longrightarrow A$. We can then define

$$\beta_* : E(A,B) \to E(A,B')$$

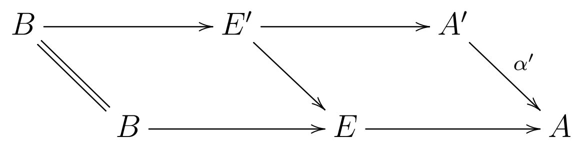
by assigning the class $B' \longrightarrow E_\beta \longrightarrow A$ to the class $B \longrightarrow E \longrightarrow A$. By the dual reasoning of the above arguments $\beta_* = E(A,B)$ makes $E(A,-)$ into a covariant functor from modules to sets.

Now consider $\beta_*\alpha^* : E(A,B) \to E(A',B')$ and $\alpha^*\beta_* : E(A,B) \to E(A',B')$. Our goal from here is to show that $\alpha^*\beta_* = \beta_*\alpha^*$.

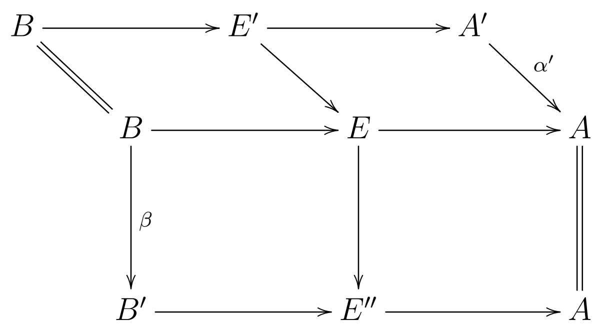
Consider the representative sequence

$$B \longrightarrow E \longrightarrow A$$

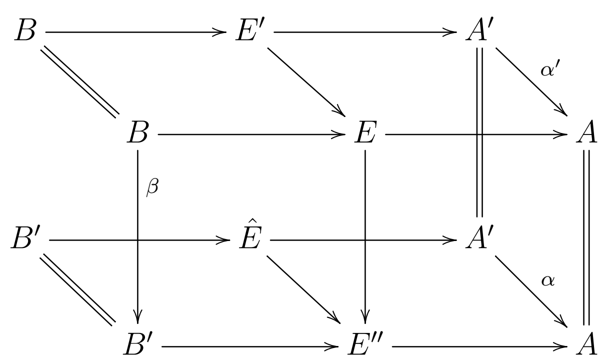
of $E(A,B)$ and module homomorphism $\alpha : A' \to A$ and $\beta : B \to B'$. Taking the pull-back on α, we get



Now taking the push-out on β we get the following diagram.



Finally, taking the pull-back of α again, we get the following 3D diagram.



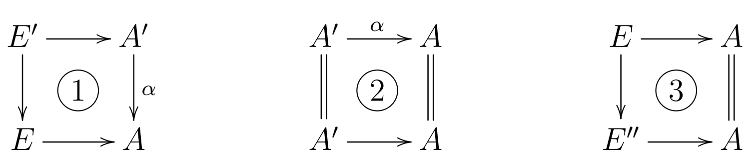
We can see that the path

$$E' \longrightarrow E \longrightarrow E'' \longrightarrow A$$

will be equivalent to

$$E' \longrightarrow A' \longrightarrow A' \xrightarrow{\alpha} A.$$

Since the squares



commute, the paths

$$E' \longrightarrow E \longrightarrow E'' \longrightarrow A$$

and

$$E' \longrightarrow E \longrightarrow A \longrightarrow A$$

are equivalent from square ③. As well as the paths

$$E' \longrightarrow E \longrightarrow A \longrightarrow A$$

and

$$E' \longrightarrow A' \longrightarrow A \longrightarrow A$$

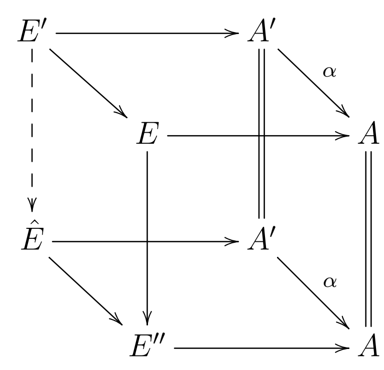
from square ②. Finally, from square ① we get the paths

$$E' \longrightarrow A' \longrightarrow A \longrightarrow A$$

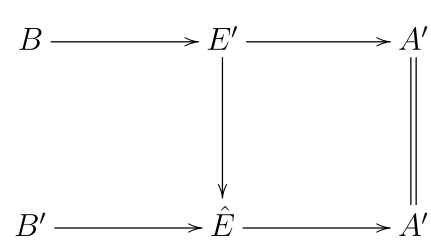
and

$$E' \longrightarrow A' \longrightarrow A' \longrightarrow A$$

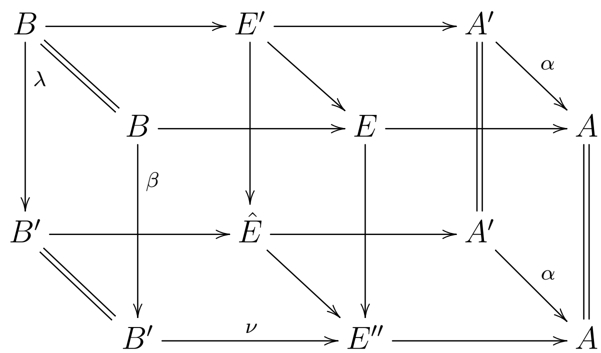
are equivalent. This would give an induced map $E' \longrightarrow \hat{E}$ making all faces on the following cube commute.



Now considering the following exact sequences with connecting maps.



There exists an induced map $\lambda : B \to B'$, making the above squares commute. This would give the following 3D diagram.



Where every face, except for possibly

$$\begin{array}{ccc} B & = & B \\ \lambda \downarrow & & \downarrow \beta \\ B' & = & B' \end{array}$$

commutes. However, from the known commutative squares

$$\begin{array}{ccc} B' & \longrightarrow & \hat{E} \\ \downarrow & (1) & \downarrow \\ B' & \longrightarrow & E'' \end{array} \qquad \begin{array}{ccc} B & \longrightarrow & E' \\ \| & (2) & \downarrow \\ B & \longrightarrow & E \end{array} \qquad \begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & (3) & \downarrow \\ \hat{E} & \longrightarrow & E'' \end{array} \qquad \begin{array}{ccc} B & \longrightarrow & E \\ \beta \downarrow & (4) & \downarrow \\ B & \xrightarrow[\nu]{} & E'' \end{array}$$

We get the path

$$B \xrightarrow{\lambda} B' \longrightarrow B' \xrightarrow{\nu} E''$$

is equivalent, by (1), to

$$B \xrightarrow{\lambda} B' \longrightarrow \hat{E} \longrightarrow E''.$$

Which, by (2), is equivalent to

$$B \longrightarrow E' \longrightarrow \hat{E} \longrightarrow E''.$$

Which by (3), is equivalent to

$$B \longrightarrow E' \longrightarrow E \longrightarrow E''.$$

Which by ② is equivalent to

$$B \longrightarrow B \longrightarrow E \longrightarrow E'',$$

and by ④, this is equivalent to

$$B \longrightarrow B \xrightarrow{\beta} B' \xrightarrow{\nu} E''.$$

Resulting in the path $B \xrightarrow{\lambda} B' \longrightarrow B'$ being equivalent to the path $B \longrightarrow B \xrightarrow{\beta} B'$. Which yields $\nu\beta = \nu\lambda$. Since ν is monic, we get that $\beta = \lambda$. Thus the square

$$\begin{CD} B @= B \\ @VVV @VVV \\ B' @= B' \end{CD}$$

commutes. By Lemma 4.4, we can see that

$$\begin{CD} B @>>> E' @>>> A' \\ @VVV @VVV @| \\ B' @>>> \hat{E} @>>> A' \end{CD}$$

is a push-out diagram. Thus $(E^{\alpha})_{\beta} = (E_{\beta})^{\alpha} = \hat{E}$, and it follows that $\beta_*\alpha^* = \alpha^*\beta_*$ being represented by

$$B' \longrightarrow \hat{E} \longrightarrow A'.$$

Resulting in the following theorem.

Theorem 4.5 E(-,-) is a bifunctor from the category of R-modules to the category of sets. It is contravariant in the first variable and covariant in the second variable.

4.2 The Functor Ext

In this section we shall develop a new bifunctor, $Ext_R(-,-)$, from the category of R-modules into the category of abelian groups and compare it with $E(-,-)$.

4.2.1 Projective Modules

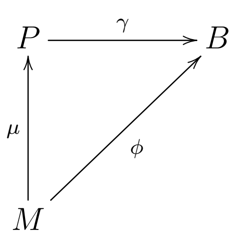
A short exact sequence $0 \longrightarrow M \xrightarrow{\mu} P \xrightarrow{\varepsilon} A \longrightarrow 0$ of R-modules with P projective is called a *projective presentation* of A. By Theorem 2.17 such a presentation induces for an R-module B an exact sequence

$$0 \longrightarrow Hom_R(A,B) \xrightarrow{\varepsilon^*} Hom_R(P,B) \xrightarrow{\mu^*} Hom_R(M,B)$$

To the modules A and B, and to the chosen projective presentation of A we can associate the abelian group

$$Ext_R^{\varepsilon}(A,B) = \operatorname{cok}\left(\mu^* : Hom_R(P,B) \to Hom_R(M,B)\right)$$

Here, $\operatorname{Im}\mu^*$ will consist of mappings, $\phi : M \to B$ that factor through P. More specifically, for $\gamma \in Hom(P,B)$, $\mu^*(\gamma) = \mu\gamma$ will give the following commutative diagram.



An element in $Ext_R^{\varepsilon}(A,B)$ may be represented by the homomorphism ϕ denoted as $[\phi]$. Then $[\phi_1] = [\phi_2]$ if and only if $\phi_1 - \phi_2$ factors through P via μ. Implying that $\phi_1 - \phi_2 = \gamma\mu$ for some γ.

Given a homomorphism $\beta : B \to B'$, create the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & Hom_R(A,B) & \xrightarrow{\varepsilon^*} & Hom_R(P,B) & \xrightarrow{\mu^*} & Hom_R(M,B) \\
& \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \beta_*} \\
0 \longrightarrow & Hom_R(A,B') & \xrightarrow{\varepsilon^*} & Hom_R(P,B') & \xrightarrow{\mu^*} & Hom_R(M,B')
\end{array}$$

Using the map $\beta_* : Hom_R(A,B) \to Hom_R(A,B')$, gives the induced map on the cokernels $\beta_* : Ext_R^{\varepsilon}(A,B) \to Ext_R^{\varepsilon}(A,B')$. Given an a exact sequence

$$0 \longrightarrow Hom(A,-) \xrightarrow{\varepsilon^*} Hom(P,-) \xrightarrow{\mu^*} Hom(M,-)$$

cok μ^* would give us the sequence

$$0 \longrightarrow Hom(A,-) \xrightarrow{\varepsilon^*} Hom(P,-) \xrightarrow{\mu^*} Hom(M,-) \longrightarrow Ext_R^{\varepsilon}(A,-) \longrightarrow 0$$

. Giving that $Ext_R^{\varepsilon}(A,-)$ is a functor.

Now, given two projective presentations of A and A', $D \xrightarrow{\mu} P \xrightarrow{\varepsilon} A$ and $D' \xrightarrow{\mu'} P' \xrightarrow{\varepsilon'} A'$, let $\alpha : A' \to A$ be a homomorphism. Since P is projective, there exists a π and then σ induced on the kernels such that

$$\begin{array}{ccccc}
D' & \longrightarrow & P' & \longrightarrow & A' \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \alpha} \\
D & \longrightarrow & P & \longrightarrow & A
\end{array}$$

commutes.

We thus get a map

$$\hat{\pi} : Ext_R^{\varepsilon}(A,B) \longrightarrow Ext_R^{\varepsilon}(A',B)$$

where the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & Hom(A,-) & \longrightarrow & Hom(P,-) & \longrightarrow & Hom(D,-) & \longrightarrow & Ext_R^{\varepsilon}(A,-) \\
& \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \sigma_*} & & \downarrow{\scriptstyle \hat{\pi}} \\
0 \longrightarrow & Hom(A',-) & \longrightarrow & Hom(P',-) & \longrightarrow & Hom(D',-) & \longrightarrow & Ext_R^{\varepsilon}(A',-)
\end{array}$$

commutes. Here for $[\phi] \in Ext^{\varepsilon}_R(A, -)$, $\hat{\pi}([\phi]) = [\phi\sigma]$. Resulting the following commutative diagram.

$$
\begin{array}{ccc}
Ext^{\varepsilon}_R(A, B) & \xrightarrow{\hat{\pi}} & Ext^{\varepsilon}_R(A', B) \\
\downarrow{\scriptstyle \hat{\beta}} & & \downarrow{\scriptstyle \hat{\beta}} \\
Ext^{\varepsilon}_R(A, B') & \xrightarrow[\hat{\pi}]{} & Ext^{\varepsilon}_R(A', B')
\end{array}
$$

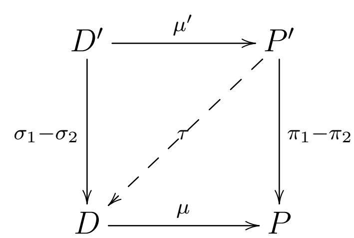
This diagram would then imply that we have a natural transformation between the functors $Ext^{\varepsilon}_R(A, -)$ and $Ext^{\varepsilon}_R(A', -)$. As well as the transformation not depending on the chosen π, only α.

Theorem 4.6 $\hat{\pi}$ does not depend on the chosen $\pi : P' \longrightarrow P$ but only on $\alpha : A' \longrightarrow A$.

Proof: Let $\pi_i : P' \to P$, $i = 1, 2$ be two homomorphisms lifting α, so that the following diagram commutes.

$$
\begin{array}{ccccc}
D' & \xrightarrow{\mu'} & P' & \xrightarrow{\varepsilon'} & A' \\
\downarrow{\scriptstyle \sigma_i} & & \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \alpha} \\
D & \xrightarrow{\mu} & P & \xrightarrow{\varepsilon} & A
\end{array}
$$

Consider $\pi_1 - \pi_2$. Since they induce the same map we have a $\tau : P' \to D$, giving the following diagram.



From the outer square, we get $\mu(\sigma_1 - \sigma_2) = (\pi_1 - \pi_2)\mu'$, and from the lifting, we get $\mu\tau = \pi_1 - \pi_2$. Thus $\mu(\sigma_1 - \sigma_2) = \mu\tau\mu'$, implying that $\sigma_1 - \sigma_2 = \tau\mu'$ and the above square commutes. This results in $\sigma_1 = \sigma_2 + \tau\mu'$. Thus if $\phi : D \to B$ is a representative of $Ext^{\varepsilon}_R(A, B)$ we have

$$\begin{aligned} \hat{\pi}_1[\phi] &= [\phi\sigma_1] \\ &= [\phi(\sigma_2 + \tau\mu')] \\ &= [\phi\sigma_2 + \phi\tau\mu'] \end{aligned}$$

Since Ext^{ε}_R operates on the cokernel of μ', we can see that $\phi\tau\mu'$ will map to [0], resulting in

$$\begin{aligned} \hat{\pi}_1[\phi] &= [\phi\sigma_2] \\ &= \hat{\pi}_2[\phi] \end{aligned}$$

∎

Let $(\alpha : P', P)$ denote the natural transformation $\hat{\pi}$. If we have three projective presentations, $D'' \longrightarrow P'' \longrightarrow A''$, $D' \longrightarrow P' \longrightarrow A'$, and $D \longrightarrow P \longrightarrow A$ with two homomorphism $\alpha' : A'' \longrightarrow A'$ and $\alpha : A' \longrightarrow A$, we can thus have two liftings $\pi' : P'' \longrightarrow P'$ and $\pi : P' \longrightarrow P$ for α' and α. Then $\pi \circ \pi' : P'' \longrightarrow P$ will lift $\alpha \circ \alpha'$ and it follows that

$$(\alpha'; P'', P') \circ (\alpha; P', P) = (\alpha \circ \alpha'; P'', P)$$

as well as

$$(1_A; P, P) = 1$$

Implying the following corollary.

Corollary 4.7 Let $D \longrightarrow P \xrightarrow{\varepsilon} A$ and $D' \longrightarrow P' \xrightarrow{\varepsilon'} A$ be two projective representations of A. Then

$$(1_A; P', P) : Ext^{\varepsilon}_R(A, -) \longrightarrow Ext^{\varepsilon'}_R(A, -)$$

is a natural equivalence.

Proof: Let $\pi : P \longrightarrow P'$ and $\pi' : P' \longrightarrow P$ both lift $1_A : A \longrightarrow A$. From the above equations, we can see that $(1_A; P, P') \circ (1_A; P', P) = (1_A; P, P) = 1$ and $(1_A; P', P) \circ (1_A; P, P') = (1_A; P, P) = 1$. Thus $(1_A; P', P)$ is a natural equivalence. ∎

By this natural equivalence we are allowed to drop ε and simply write $Ext_R(A, B)$.

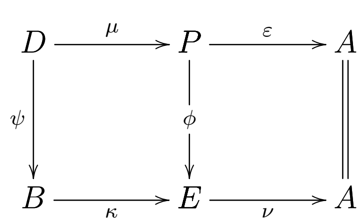
Now given $\alpha : A' \longrightarrow A$ we can define an $\hat{\alpha}$ as follows. Choosing two projective presentations $D' \longrightarrow P' \longrightarrow A'$ and $D \longrightarrow P \longrightarrow A$, we let $\hat{\alpha} = \hat{\pi} = (\alpha; P', P) : Ext_R(A, B) \longrightarrow Ext_R(A', B)$. Which we can see that by this definition of $\hat{\alpha}$ we line up with Corollary 4.7 giving $Ext_R(-, B)$ is a covariant functor. Implying the following theorem.

Theorem 4.8 $Ext_R(-, -)$ is a bifunctor from the category of R-modules into the category of abelian groups. It is contravariant in the first variable and covariant in the second.

Since the category of abelian groups is a subcategory of **Set** we can regard $Ext_R(-, -)$ as a **Set**-valued bifunctor.

Theorem 4.9 There is a natural equivalence of **Set**-valued bifunctors $\eta : E(-, -) \longrightarrow Ext_R(-, -)$.

Proof: Given an element in $E(A, B)$, represented by the extension $B \xrightarrow{\kappa} E \xrightarrow{\nu} A$ we form the following commutative diagram.



The homomorphism $\psi : D \longrightarrow B$ defines an element $[\psi] \in Ext_R(A, B)$. Let $\phi_i : P \longrightarrow E$, $i = 1, 2$ be two maps inducing $\psi_i : D \longrightarrow B$. Then $\phi_1 - \phi_2$ factors through $\tau : P \longrightarrow B$, or $\phi_1 - \phi_2 = \kappa\tau$. It follows that $\psi_1 - \psi_2 = \tau\mu$, giving: $[\psi_1] = [\psi_2 + \tau\mu] = [\psi_2]$. Thus the particular ψ is independent of the ϕ chosen.

Given two representatives of the same element in $E(A, B)$, we can create the following commutative diagram.

$$\begin{array}{ccccc}
B & \longrightarrow & E & \longrightarrow & A \\
\downarrow{\scriptstyle \beta} & & \downarrow & & \downarrow{\scriptstyle \alpha} \\
B & \longrightarrow & E' & \longrightarrow & A
\end{array}$$

Where α and β are both isomorphisms. Implying that the projective presentations of A will be isomorphic. It follows that two representatives of the same element in $E(A, B)$ will induce the same element in $Ext^{\varepsilon}_R(A, B)$. Thus we have a well-defined $\eta : E(A, B) \to Ext^{\varepsilon}_R(A, B)$.

Now given an element in $Ext_R(A, B)$, we represent this element by a homomorphism $\psi : D \longrightarrow B$. Taking the push-out of (ψ, μ) we obtain the following commutative diagram.

$$\begin{array}{ccccc}
D & \xrightarrow{\mu} & P & \xrightarrow{\varepsilon} & A \\
{\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \phi} & & \| \\
B & \xrightarrow[\kappa]{} & E & \xrightarrow[\nu]{} & A
\end{array}$$

By the dual of Lemma 4.2 the bottom row is an extension. Given another representative $\psi' : D \longrightarrow B$ of the form $\psi' = \psi + \tau\mu$, where $\tau : P \longrightarrow B$, we are given the following diagram.

$$\begin{array}{ccccc} D & \xrightarrow{\mu} & P & \xrightarrow{\varepsilon} & A \\ \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \phi'} & & \| \\ B & \xrightarrow[\kappa]{} & E & \xrightarrow[\nu]{} & A \end{array}$$

with $\phi' = \phi + \kappa\tau$. We wish this to be commutative, or $\phi'\mu = \kappa\psi'$.

$$\begin{array}{rcl} \psi' & = & \psi + \tau\mu \\ \kappa\psi' & = & \kappa\psi + \kappa\tau\mu \\ & = & \phi\mu + \kappa\tau\mu \\ & = & (\phi + \kappa\tau)\mu \\ & = & \phi'\mu \end{array}$$

By Lemma 4.4 the left hand square is a push-out diagram. Meaning, given any representative, we can create the original ψ and thus the extension is not dependent on the representative. Giving a well-defined map

$$\xi : Ext_R(A, B) \longrightarrow E(A, B).$$

Which is also natural in B. Given a representative of $E(A, B)$ η will give a $\psi : D \longrightarrow B$ such that

$$\begin{array}{ccccc} D & \longrightarrow & P & \longrightarrow & A \\ \downarrow & & \downarrow & & \| \\ B & \longrightarrow & E & \longrightarrow & A \end{array}$$

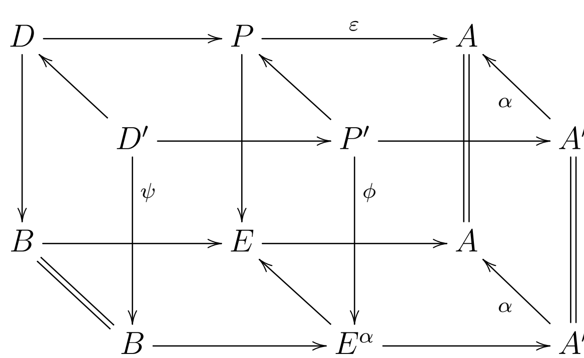
commutes. By Lemma 4.4, we get the left hand square is a pull-back. Which is unique up to isomorphism. Implying that ξ will return the same extension.

Conversely, given $\psi : D \longrightarrow B$ ξ will give us the extension

$$\begin{array}{ccccc} D & \longrightarrow & P & \longrightarrow & A \\ \downarrow & & \downarrow & & \| \\ B & \longrightarrow & E & \longrightarrow & A \end{array}$$

Again by Lemma 4.4, we will get that η will give back ψ, and η and ξ are inverses of each other. Giving an equivalence $\eta : E(A, B) \longrightarrow Ext_R(A, B)$.

Note that it may be possible that η is dependent upon our choice of projective presentation. However, we show this is not the case with the following diagram.



E^α is the pull-back of $E \longrightarrow A$ and $A' \longrightarrow A$. We need to show the existence of homomorphisms, $\phi : P' \longrightarrow E^\alpha$ and $\psi : D' \longrightarrow B$ such that this diagram commutes. Since $P' \longrightarrow E \longrightarrow A$ and $P' \longrightarrow A' \longrightarrow A$ agree they define a homomorphism $\phi : P' \longrightarrow E^\alpha$ into the pull-back. The ϕ then induces $\psi : D' \longrightarrow B$, resulting in all faces commutative. Meaning if given a representative in $E(A, B)$, η will give us a corresponding ψ, and if $\hat{\alpha}$ send this representative to another extension we can find another corresponding ψ'. We then arrive at the following diagram:

$$\begin{array}{ccc} E(A,B) & \xrightarrow{\hat{\alpha}} & E(A',B) \\ {\scriptstyle\xi}\big\uparrow\big\downarrow{\scriptstyle\eta} & & {\scriptstyle\xi}\big\uparrow\big\downarrow{\scriptstyle\eta} \\ Ext_R(A,B) & \xrightarrow{\hat{\alpha}} & Ext_R(A',B) \end{array}$$

For $A' = A$, $\alpha = 1_A$ this shows the independence of the chosen presentation.

Furthermore, by similar arguments, we can create the following commutative diagram

$$\begin{array}{ccc} E(A,B) & \xrightarrow{\hat{\beta}} & E(A,B') \\ {\scriptstyle\xi}\big\uparrow\big\downarrow{\scriptstyle\eta} & & {\scriptstyle\xi}\big\uparrow\big\downarrow{\scriptstyle\eta} \\ Ext_R(A,B) & \xrightarrow{\hat{\beta}} & Ext_R(A,B') \end{array}$$

for the naturality in B. Thus concluding the proof. ∎

Corollary 4.10 The set $E(A, B)$ has an abelian structure.

Proof: This is shown by Theorem 4.9 and the fact that $Ext_R(A, B)$ has an abelian structure. ∎

We will now investigate the structure on $E(A, B)$. Given two representatives of $E(A, B)$,

$$B \longrightarrow E_i \longrightarrow A$$

for $i = 1, 2$. We will first consider the direct sum.

$$B \oplus B \longrightarrow E_1 \oplus E_2 \longrightarrow A \oplus A$$

Define a map $\Delta_A : A \longrightarrow A \oplus A$ by $\Delta_A(a) = (a, a)$. Since addition is defined componentwise, Δ_A is clearly a module homomorphism. Let F be the pullback of Δ_A and $E_1 \oplus E_2 \longrightarrow A \oplus A$. Giving the following diagram.

$$\begin{array}{ccccc} B \oplus B & \longrightarrow & F & \longrightarrow & A \\ \big\| & & \big\downarrow & & \big\downarrow{\scriptstyle\Delta_A} \\ B \oplus B & \longrightarrow & E_1 \oplus E_2 & \longrightarrow & A \oplus A \end{array}$$

Now define a map $\nabla_B : B \oplus B \longrightarrow B$ by $\nabla_B(b_1, b_2) = b_1 + b_2$ for $b_1, b_2 \in B$. Which is also clearly a module homomorphism. Let E be the push-out of ∇_B and $B \oplus B \longrightarrow F$. Resulting in the following diagram.

$$\begin{array}{ccccc} B \oplus B & \longrightarrow & F & \longrightarrow & A \\ \downarrow{\scriptstyle \nabla_B} & & \downarrow & & \| \\ B & \longrightarrow & E & \longrightarrow & A \end{array}$$

Where $B \longrightarrow E \longrightarrow A$ is the sum of the two representatives.

Example 4.11 Adding the elements of $Ext(\mathbb{Z}_2, \mathbb{Z}_4)$

Recall that $Ext(\mathbb{Z}_2, \mathbb{Z}_4)$ contained the elements $\mathbb{Z}_4 \xrightarrow{\nu} \mathbb{Z}_8 \xrightarrow{\pi_1} \mathbb{Z}_2$ and $\mathbb{Z}_4 \xrightarrow{i} \mathbb{Z}_4 \oplus \mathbb{Z}_2 \xrightarrow{\pi_2} \mathbb{Z}_2$ where ν is multiplication by 2, i is the injection map, and each π is the projection map. Now consider the direct sum of these sequences

$$\mathbb{Z}_4 \oplus \mathbb{Z}_4 \xrightarrow{\gamma} \mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \xrightarrow{\hat{\pi}} \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

with $\gamma = [\nu, i]$ and $\hat{\pi} = [\pi_1, \pi_2]^T$. Let F be the pull-back of $\Delta_{\mathbb{Z}_2}$ and $\hat{\pi}$. Since $F = \{(a, b) | \Delta_{\mathbb{Z}_2}(a) = \hat{\pi}(a, (b_1, b_2))\}$, this would then give $F = \mathbb{Z}_8 \oplus \mathbb{Z}_2$. Now let E be the push-out of $\nabla_{\mathbb{Z}_4}$ and γ. Recall that $E = (\mathbb{Z}_8 \oplus \mathbb{Z}_2)/S$ where $S = \{(\nabla_{\mathbb{Z}_4}(b_1, b_2), -\gamma(b_1, b_2)\}$. Thus $E = \mathbb{Z}_8$, resulting in $\mathbb{Z}_4 \longrightarrow \mathbb{Z}_8 \longrightarrow \mathbb{Z}_2$ as the sum. □

We can actually find the sum of two representatives of $E(A, B)$ via projectives as follows. Given two representatives of $E(A, B)$ we can create the following push-out diagram for $i = 1, 2$.

$$\begin{array}{ccccc} K & \longrightarrow & P & \longrightarrow & A \\ \downarrow{\scriptstyle \lambda_i} & & \downarrow & & \| \\ B & \longrightarrow & E_i & \longrightarrow & A \end{array}$$

The sum can be computed by the following diagram.

$$\begin{array}{ccccc} K & \longrightarrow & P & \longrightarrow & A \\ \downarrow{\scriptstyle \lambda_1+\lambda_2} & & \downarrow & & \| \\ B & \longrightarrow & E & \longrightarrow & A \end{array}$$

We finally note:

Theorem 4.12 If P is projective and I is injective, then $Ext_R(P, B) = 0 = Ext_R(A, I)$.

Proof: Theorem 4.9 gives us that $Ext_R(A, B)$ has a one-to-one correspondence with the set $E(A, B)$. Elements in $E(A, B)$ are extensions of the form $B \longrightarrow E \longrightarrow A$. If $A = P$, then by Theorem 2.28 the sequence splits. Similarly, if $B = I$ the sequence splits by the dual of Theorem 2.28. If $B \longrightarrow E \longrightarrow A$ splits, then $E(A, B)$ contains only one element, namely 0. ∎

4.2.2 Injective Modules

Here we will consider the dual argument to create $\overline{Ext}^{\nu}_R(A, B)$ for injective presentations $B \longrightarrow I \longrightarrow S$. One could develop all of the same (but dual) theory as in the previous section to create $\overline{Ext}^{\nu}_R(A, B)$ as a functor and set up the natural equivalence to $E(-, -)$. However, it will be advantageous to develop $\overline{Ext}^{\nu}_R(A, B)$ in a different way to show more about the structure of $E(-, -)$.

First we introduce new notation to shorten the arguments. Define Σ to be a commutative diagram of R-modules

$$\begin{array}{ccc} A' & \xrightarrow{\alpha} & A \\ \Big\downarrow{\scriptstyle \psi} & \Sigma & \Big\downarrow{\scriptstyle \phi} \\ B & \xrightarrow[\beta]{} & B' \end{array}$$

and

$$\operatorname{Im}\Sigma = \operatorname{Im}\phi \cap \operatorname{Im}\beta / \operatorname{Im}\phi\alpha$$

$$\operatorname{Ker}\Sigma = \ker\phi\alpha \ /(\ker\alpha + \ker\psi)$$

For $\operatorname{Im}\Sigma$ to be well defined we must have $\operatorname{Im}\phi\alpha \subseteq \operatorname{Im}\phi \cap \operatorname{Im}\beta$. Suppose that $x \in \operatorname{Im}\phi\alpha$. This would imply that $x \in \operatorname{Im}\phi$. By commutativity, $\phi\alpha = \beta\psi$. Resulting in $\operatorname{Im}\phi\alpha \subseteq \operatorname{Im}\phi \cap \operatorname{Im}\beta$.

For $\operatorname{Ker}\Sigma$ to be will defined we need $\ker\alpha + \ker\psi \subseteq \ker\phi\alpha$. Suppose that $x \in \ker\alpha + \ker\psi$. This would mean that $x = a + b$ such that $a \in \ker\alpha$ and $b \in \ker\psi$. Thus

$$\begin{array}{rcl} x = & a + b & \\ \phi\alpha(x) = & \phi\alpha(a+b) & \\ = & \phi\alpha(a) + \phi\alpha(b) & \phi \text{ and } \alpha \text{ are homomorphisms} \\ = & \phi\alpha(a) + \beta\psi(b) & \text{commutativity of } \Sigma \\ = & \phi(0) + \beta(0) & \\ = & 0 & \end{array}$$

Giving $\ker\alpha + \ker\psi \subseteq \ker\phi\alpha$.

Lemma 4.13 Let

$$\begin{array}{ccccc}
A' & \xrightarrow{\alpha_1} & A & \xrightarrow{\alpha_2} & A'' \\
\Big\downarrow{\scriptstyle \psi} & \Sigma_1 & \Big\downarrow{\scriptstyle \phi} & \Sigma_2 & \Big\downarrow{\scriptstyle \theta} \\
B' & \xrightarrow{\beta_1} & B & \xrightarrow{\beta_2} & B''
\end{array}$$

be a diagram with exact rows, then ϕ induces an isomorphism $\Phi : \mathrm{Ker}\,\Sigma_2 \xrightarrow{\backsim} \mathrm{Im}\,\Sigma_1$.

Proof: Let$x \in \ker\theta\alpha_2$, clearly $\phi(x) \in \mathrm{Im}\,\phi$. Since $(\theta\alpha_2)(x) = (\beta_2\phi)(x) = 0$, $\phi(x) \in \mathrm{Im}\,\beta_1$. Thus we can define $\Phi(x) = \phi(x)$. Now let $x \in \ker\alpha_2$, so $x = a_1 + a_2$ with $\alpha_2(a_1) = 0$ and $\phi(a_2) = 0$. Thus

$$\begin{aligned}
\Phi(x) &= \phi(a_1) + \phi(a_2) \\
&= \phi(a_1) \\
&= \phi\alpha_1(a') \\
&= \overline{0}
\end{aligned}$$

Resulting in Φ as well-defined. Now let $y \in \mathrm{Im}\,\phi \cap \mathrm{Im}\,\beta_1$. Then there exists an $x \in A$ such that $\phi(x) = y$. By commutativity, we get $(\theta\alpha_2)(x) = (\beta_2\phi)(x) = \beta_2(y)$. since $y \in \mathrm{Im}\,\beta_1$ and the bottom row is exact, we find that $\beta_2(y) = 0$ and $x \in \ker\theta\alpha_2$ and Φ is onto. Finally, suppose that $x \in \ker\theta\alpha_2$ such that $\phi(x) \in \mathrm{Im}\,\phi\alpha_1$, or $\phi(x) = (\phi\alpha_1)(z)$ for some $z \in A'$. From the exactness of the top row, we have $\alpha_1(z)\ker\alpha_2$. Now $\phi(x - \alpha_1(z)) = 0$, so let $t = x - \alpha_1(z)$, where $t \in \ker\phi$. Thus $x = \alpha_1(z) + t$. Implying that $x \in \ker\alpha_2 + \ker\phi$. Making Φ one-to-one and thus an isomorphism. ∎

Theorem 4.14 For any projective presentation $D \xrightarrow{\mu} P \xrightarrow{\varepsilon} A$ of A and injective presentation $B \xrightarrow{\nu} I \xrightarrow{\eta} S$ of B there is an isomorphism

$$\sigma : Ext_R^{\varepsilon}(A, B) \longrightarrow \overline{Ext}_R^{\nu}(A, B)$$

Proof: Consider the following commutative diagram with exact columns and rows.

$$\begin{array}{ccccccc}
Hom_R(A,B) & \longrightarrow & Hom_R(A,I) & \longrightarrow & Hom_R(A,S) & \longrightarrow & \overline{Ext}^{\nu}_R(A,B) \\
\downarrow & & \downarrow & \Sigma_2 & \downarrow & \Sigma_1 & \downarrow \\
Hom_R(P,B) & \longrightarrow & Hom_R(P,I) & \longrightarrow & Hom_R(P,S) & \longrightarrow & 0 \\
\downarrow & \Sigma_4 & \downarrow & \Sigma_3 & \downarrow & & \\
Hom_R(D,B) & \longrightarrow & Hom_R(D,I) & \longrightarrow & Hom_R(D,S) & & \\
\downarrow & \Sigma_5 & \downarrow & & & & \\
Ext^{\varepsilon}_R(A,B) & \longrightarrow & 0 & & & &
\end{array}$$

Looking at Σ_1, we have the following commutative diagram.

$$\begin{array}{ccc}
Hom_R(A,S) & \xrightarrow{\alpha} & \overline{Ext}^{\nu}_R(A,B) \\
\psi \downarrow & \Sigma_1 & \downarrow \phi \\
Hom_R(P,S) & \xrightarrow[\beta]{} & 0
\end{array}$$

Since ψ is injective $\ker \psi = 0$ giving $\operatorname{Ker} \Sigma_1 = \ker \phi\alpha / \ker \alpha = \ker \phi = \overline{Ext}^{\nu}_R(A,B)$.

Now looking at Σ_5, we have this commutative diagram.

$$\begin{array}{ccc}
Hom_R(D,B) & \xrightarrow{\alpha} & Hom_R(D,I) \\
\psi \downarrow & \Sigma_5 & \downarrow \phi \\
Ext^{\varepsilon}_R(A,B) & \xrightarrow[\beta]{} & 0
\end{array}$$

Here $\ker \phi\alpha = Hom_R(D,B)$ and ψ is onto, meaning that for each $\overline{x} \in \operatorname{Ker} \Sigma_5$ there exists a $y \in Hom_R(D,B)$ such that $\psi(y) = x$. Giving $\operatorname{Ker} \Sigma_5 = Ext^{\varepsilon}_R(A,B)$.

Now applying Lemma 4.13 several times we find

$$\overline{Ext}^{\nu}_R(A,B) = \operatorname{Ker} \Sigma_1 \cong \operatorname{Im} \Sigma_2 \cong \operatorname{Ker} \Sigma_3 \cong \operatorname{Im} \Sigma_4 \cong \operatorname{Ker} \Sigma_5 = Ext^{\varepsilon}_R(A,B)$$

∎

Thus for any injective presentation of B, $\overline{Ext}^{\nu}_R(A,B)$ is isomorphic to $Ext_R(A,B)$, allowing us to drop the ν. Now let $\beta : B \longrightarrow B'$ be a homomorphism and let $B' \xrightarrow{\nu'} I' \longrightarrow S'$ be an injective presentation. If $\tau : I \longrightarrow I'$ is a map that induces β, we then obtain a corresponding diagram as seen in the proof of Theorem 4.14. Resulting in the induced homomorphism

$$\beta_* : \overline{Ext}_R(A,B) \longrightarrow \overline{Ext}_R(A,B')$$

Which agrees with the above isomorphism and the homomorphism
$\beta_* : Ext_R(A,B) \longrightarrow Ext_R(A,B')$. Following an analogous procedure, we can define $\alpha^* : \overline{Ext}_R(A,B) \longrightarrow \overline{Ext}_R(A',B)$. Using these definitions we get the following theorem.

Theorem 4.15 $\overline{Ext}_R(-,-)$ is a bifunctor, contravariant in the first variable and covariant in the second. It is naturally equivalent to $Ext_R(-,-)$ and therefore naturally equivalent to $E(-,-)$.

Since all of these functor are equivalent only one notation is traditionally used, $Ext_R(-,-)$.

Example 4.16 For the following examples the ground ring will be the integers.

- Since $\mathbb{Z}$ is projective, Theorem 4.12 gives

$$Ext(\mathbb{Z},\mathbb{Z}) = 0 = Ext(\mathbb{Z},\mathbb{Z}_q)$$

- $Ext(\mathbb{Z}_r,\mathbb{Z})$

  Using the projective presentation $\mathbb{Z} \xrightarrow{\mu} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}_r$ where μ is multiplication by r. This gives the exact sequence

$$\begin{array}{ccccccc}
Hom(\mathbb{Z}_r,\mathbb{Z}) & \longrightarrow & Hom(\mathbb{Z},\mathbb{Z}) & \xrightarrow{\mu^*} & Hom(\mathbb{Z},\mathbb{Z}) & \longrightarrow & Ext(\mathbb{Z}_r,\mathbb{Z}) \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\mu^*} & \mathbb{Z} & &
\end{array}$$

since μ^* is multiplication by r, we get

$$Ext(\mathbb{Z}_r,\mathbb{Z}) \cong \mathbb{Z}_r$$

- $Ext(\mathbb{Z}_r,\mathbb{Z}_q)$

From Example 2.5 we know that $Hom(\mathbb{Z}_r,\mathbb{Z}_q) \cong \mathbb{Z}_{(r,q)}$, where (r,q) is the greatest common divisor of q and r. Giving us the following the following diagram.

$$\begin{array}{ccccccc}
Hom(\mathbb{Z}_r,\mathbb{Z}_q) & \longrightarrow & Hom(\mathbb{Z},\mathbb{Z}_q) & \xrightarrow{\mu^*} & Hom(\mathbb{Z},\mathbb{Z}_q) & \longrightarrow & Ext(\mathbb{Z}_r,\mathbb{Z}_q) \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\
\mathbb{Z}_{(q,r)} & \longrightarrow & \mathbb{Z}_q & \xrightarrow{\mu^*} & \mathbb{Z}_q & &
\end{array}$$

Where μ^* is defined as multiplication by r. Here $\operatorname{cok}\mu^* = \mathbb{Z}_q/r\mathbb{Z}_q \cong \mathbb{Z}_{(r,q)}$. Which gives $Ext(\mathbb{Z}_r,\mathbb{Z}_q) \cong \mathbb{Z}_{(r,q)}$.

□

Chapter 5

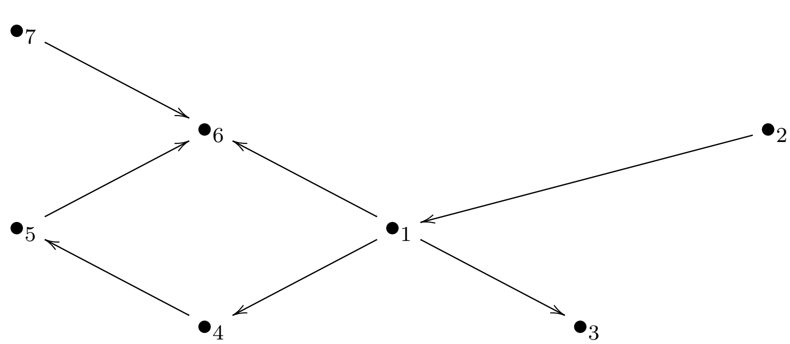
Ext in RepG

Here we will compute the functor Ext for a representations of simple digraphs. We will begin by discussing basic results in **Rep**G to aid in finding the structure of Ext.

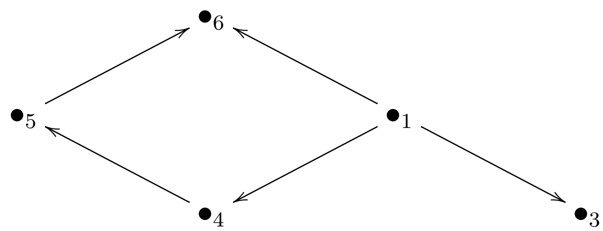
5.1 Projectives in RepG

For a given digraph G, we will pick any vertex in G and consider all the paths which start from there.

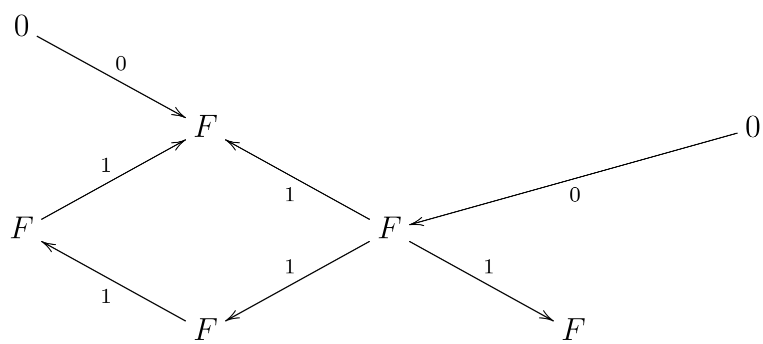
Example 5.1



We will form the subgraph L be starting from the vertex 1. Giving us the graph:



Here we will assign each vertex in L the one dimensional vector space F and each edge in L the identity linear transformation, denoted by 1. For all other elements we will label them 0. Making the digraph $\mathscr{P}(1)$.

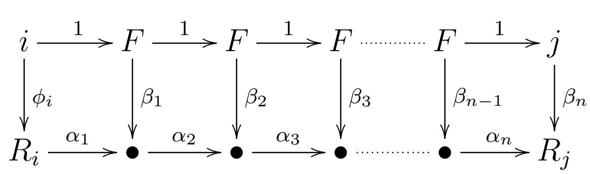


∎

Lemma 5.2 Given the digraph G has no cycles or multiple edges between two vertices, then $Hom(\mathscr{P}(i), R) \cong R_i$ as abelian groups.

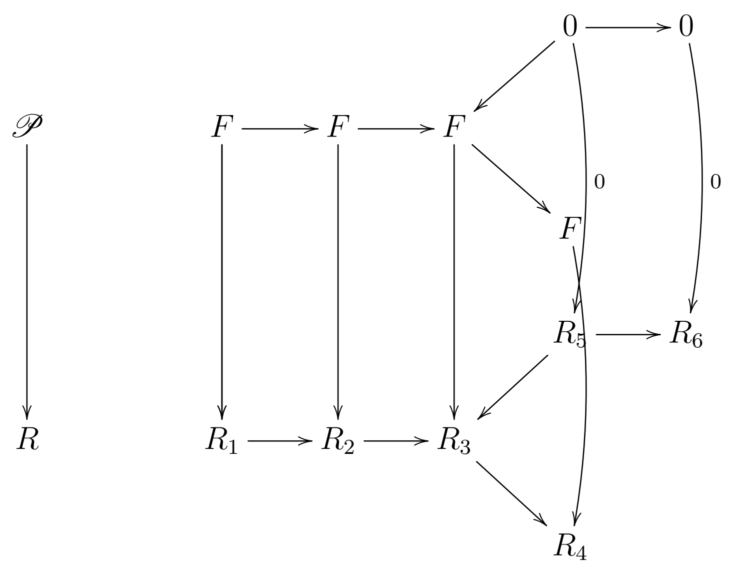
Proof: Suppose $\phi : \mathscr{P}(i) \to R$, in particular $\phi_i : F \to R$ and $\phi_i(1) \in R_i$. This gives a map $Hom(\mathscr{P}(i), R) \to R_i$.

For the inverse, let $v \in R_i$. This defines $\phi_i : F \to R_i$ to be $\phi_i(\lambda) = \lambda v$. Suppose $\mathscr{P}(j) \neq 0$, so there exists a path from i to j. This would result in the following commutative diagram.



In order for the β_i to give a commute diagram, one require $\beta_1 = \alpha_1\phi_i$, $\beta_2 = \alpha_2\alpha_1\phi_i$,...,$\beta_n = \alpha_n\alpha_{n-1}...\alpha_1\phi_i$, resulting in a unique set of linear transformations. For any vertex not in such a path use the zero transformation. Hence each $v \in R_i$ yields a unique morphism in $Hom(\mathscr{P}(i), R)$ and it follows that $R_i \cong Hom(\mathscr{P}(i), R)$ as vector spaces. Since if v and w yield ϕ_v and ϕ_w in $Hom(\mathscr{P}, R)$, then $\lambda v + w$ will yield $\lambda\phi_v + \phi_w$. ∎

Example 5.3



□

Theorem 5.4 Assume G has no multiple edges between two vertices and no cycles. Then the previous construction $\mathscr{P}(i)$, for any vertex i, is a projective object in **Rep** G.

Proof: Now given two presentations R and S with $R \longrightarrow S$ onto and $\mathscr{P}(i) \longrightarrow S$. Since $R_i \longrightarrow S_i$ is onto, pick w such that $w \mapsto v$, $w \in R_i$ and $v \in S_i$ with $1 \to v$ for $1 \in \mathscr{P}(i)$. Thus $w \in R_i$ and by the above lemma yields linear transformation $\phi \in \mathscr{P}(i) \longrightarrow R$. In the compositions $\mathscr{P}(i) \longrightarrow R_i \longrightarrow S_i$ we get

$1 \longrightarrow w \longrightarrow v$, so, by the above lemma again, composition must be the original $\mathscr{P}(i) \longrightarrow S$. ∎

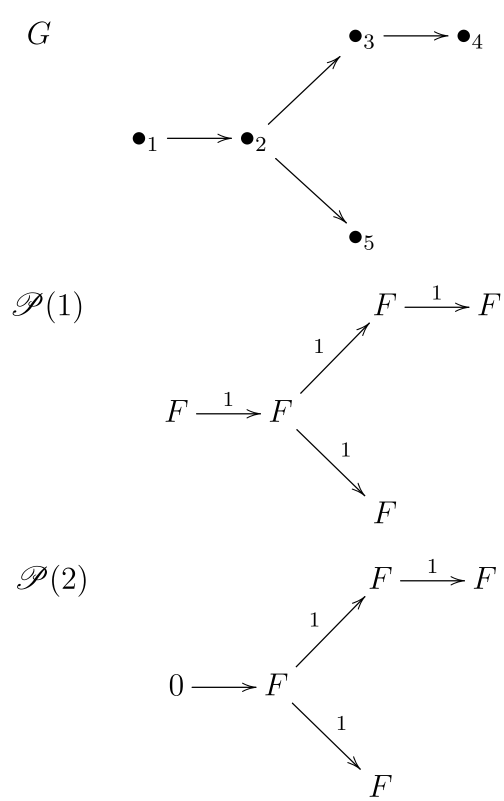
Example 5.5

G $\qquad \bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3$

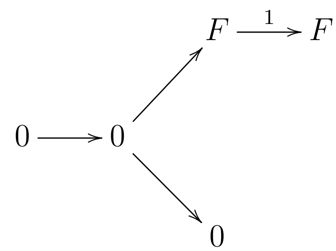
$\mathscr{P}(1)$ $\qquad F \xrightarrow{1} F \xrightarrow{1} F$

$\mathscr{P}(2)$ $\qquad 0 \longrightarrow F \xrightarrow{1} F$

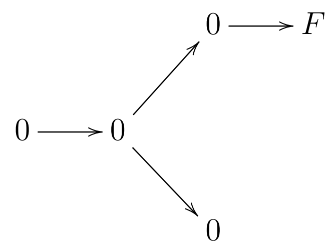
$\mathscr{P}(3)$ $\qquad 0 \longrightarrow 0 \longrightarrow F$



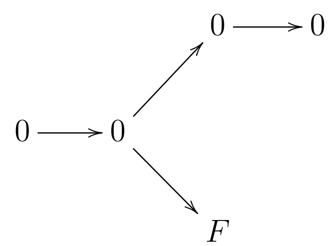
$\mathscr{P}(3)$



$\mathscr{P}(4)$

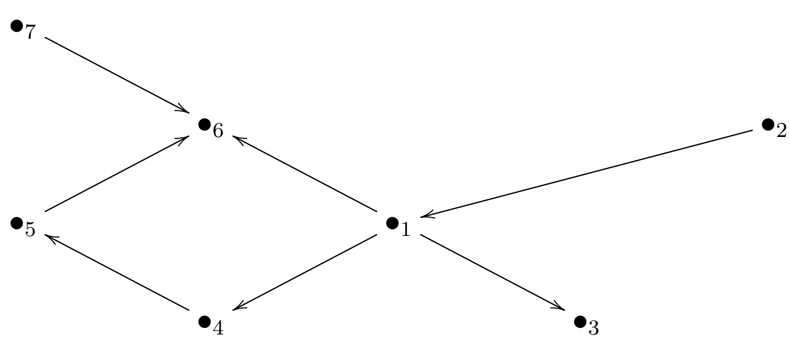


$\mathscr{P}(5)$

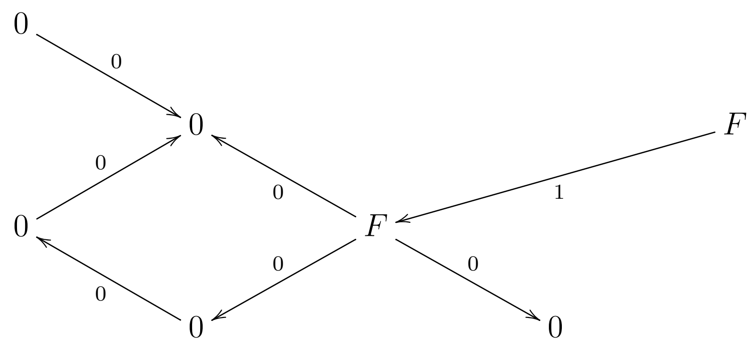


∎

For the dual notion of injective presentations in **Rep** G we use a similar construction but with paths ending at i. So with G



will give the following injective presentation $\mathscr{I}(1)$



A simpler example would be

G $\qquad \bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3$

$\mathscr{I}(1) \qquad F \longrightarrow 0 \longrightarrow 0$

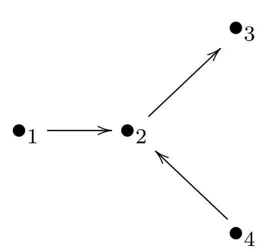
$\mathscr{I}(2) \qquad F \xrightarrow{1} F \longrightarrow 0$

$\mathscr{I}(3) \qquad F \xrightarrow{1} F \xrightarrow{1} F$

Now given a representation R and any vertex i, let V_i be the vector space at i. This can be expressed as a direct sum of the field F. Thus assigning V_i at i and at any other vertex reachable from i using the identity transformation. For all paths starting at i we get a direct sum of copies of $\mathscr{P}(i)$ and thus will be projective. We will denote this as $\mathscr{P}(V_i)$, and then $\oplus\mathscr{P}(V_i)$, the sum over the vertices, will also be projective and maps onto R.

Example 5.6

G



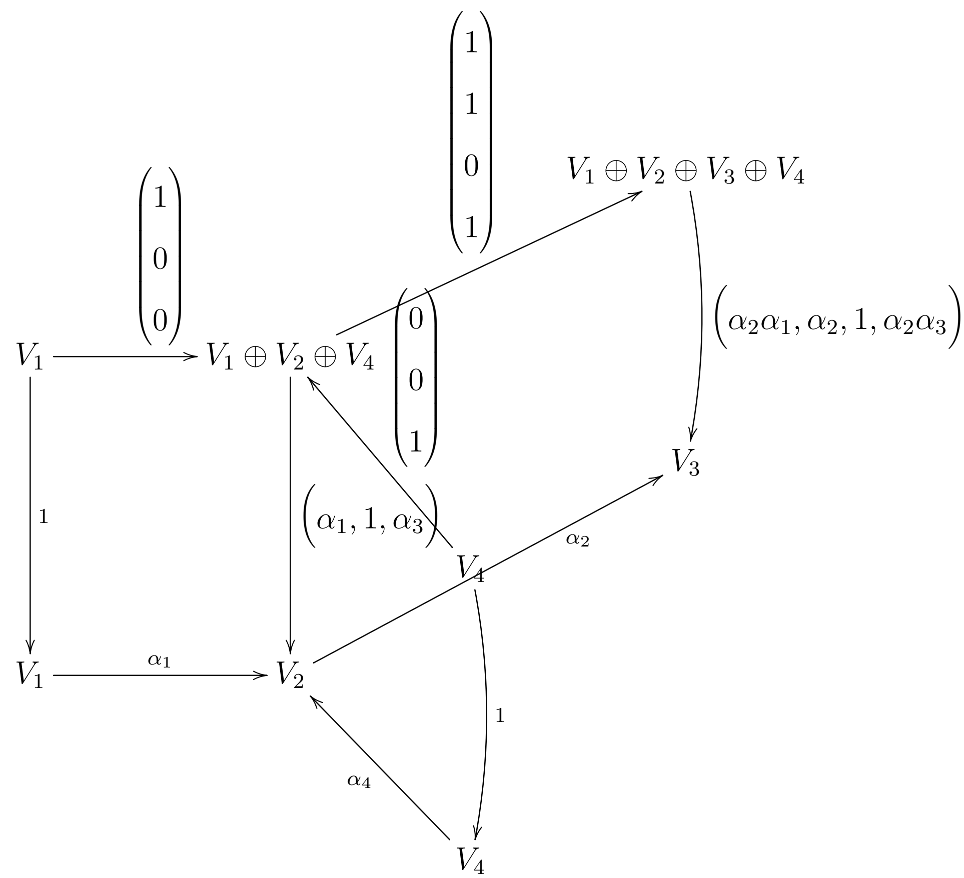
□

This would result in the following.

$$\mathscr{P}(V_1) \qquad \begin{array}{ccccc} & & & & V_1 \\ & & & \nearrow^{1} & \\ V_1 & \xrightarrow{1} & V_1 & & \\ & & & \nwarrow & \\ & & & & 0 \end{array} \qquad\qquad \mathscr{P}(V_2) \qquad \begin{array}{ccccc} & & & & V_2 \\ & & & \nearrow^{1} & \\ 0 & \longrightarrow & V_2 & & \\ & & & \nwarrow & \\ & & & & 0 \end{array}$$

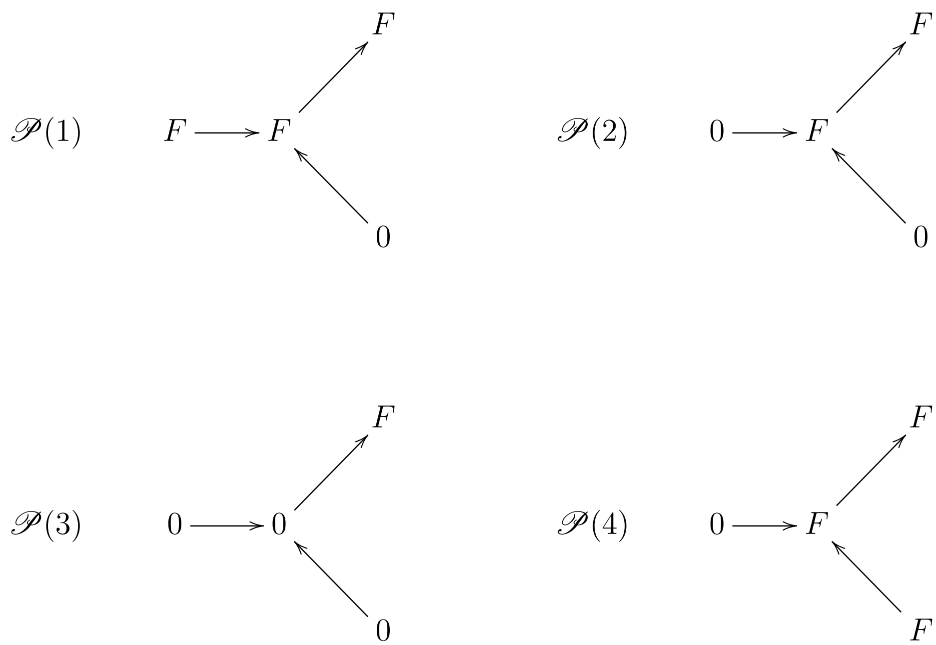
$$\mathscr{P}(V_3) \qquad \begin{array}{ccccc} & & & & V_3 \\ & & & \nearrow & \\ 0 & \longrightarrow & 0 & & \\ & & & \nwarrow & \\ & & & & 0 \end{array} \qquad\qquad \mathscr{P}(V_4) \qquad \begin{array}{ccccc} & & & & V_4 \\ & & & \nearrow^{1} & \\ 0 & \longrightarrow & V_4 & & \\ & & & \nwarrow_{1} & \\ & & & & V_4 \end{array}$$

Which would give the following map.



This result is onto since at each vertex i, $\mathscr{P}(V_i) \longrightarrow R_i$ was onto. Implying that any projective presentation is the direct sum of copies of $\mathscr{P}(i)$'s.

Applying Lemma 5.2, we can see that $Hom(\mathscr{P}(i), \mathscr{P}(j)) = F$ if there is a path from i to j, otherwise $Hom(\mathscr{P}(i), \mathscr{P}(j)) = 0$. Continuing with the previous example we have,



Resulting in

$$\begin{aligned}
Hom(\mathscr{P}(1), \mathscr{P}(1)) &= F \\
Hom(\mathscr{P}(1), \mathscr{P}(2)) &= F \\
Hom(\mathscr{P}(1), \mathscr{P}(3)) &= F \\
Hom(\mathscr{P}(1), \mathscr{P}(4)) &= 0
\end{aligned}$$

We can express this as a matrix where the $(i, j)-$entry is $Hom(\mathscr{P}(i), \mathscr{P}(j))$. This would result in the following matrix.

$$\begin{bmatrix} F & F & F & 0 \\ 0 & F & F & 0 \\ 0 & 0 & F & 0 \\ 0 & F & F & F \end{bmatrix}$$

In general, if G has n vertices with the graph being a tree (i.e. no cycle) the result would be an $n \times n$ matrix with entries 0 and F. This matrix represents all the linear transformations of $\mathscr{P}(1) \oplus \mathscr{P}(2) \oplus \cdots \oplus \mathscr{P}(n) = \mathscr{P}$ onto itself. In other words $End(\mathscr{P})$. Since F is a field, we can express this as an $n \times n$ matrix with entries from F. Thus

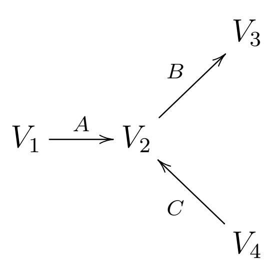
$$End(\mathscr{P}) \cong \{[a_{ij}] | a_{ij} = Hom(\mathscr{P}(i), \mathscr{P}(j))\}$$

Identifying $Hom(\mathscr{P}(i), \mathscr{P}(j))$ with F and a_{ij} with $b_{ij} \in F$, then $b_{ij} = 0$ if $Hom(\mathscr{P}(i), \mathscr{P}(j)) = 0$. So continuing our example

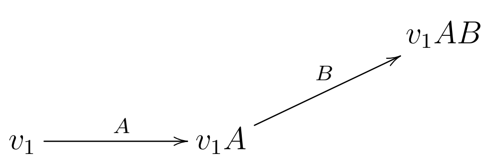
$$\begin{aligned} End(\mathscr{P}(1) \oplus \mathscr{P}(2) \oplus \mathscr{P}(3) \oplus \mathscr{P}(4) \quad &\cong \begin{bmatrix} F & F & F & 0 \\ 0 & F & F & 0 \\ 0 & 0 & F & 0 \\ 0 & F & F & F \end{bmatrix} \\ &= \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ 0 & a_4 & a_5 & 0 \\ 0 & 0 & a_6 & 0 \\ 0 & a_7 & a_8 & a_9 \end{bmatrix} | a_i \in F \ all \ i \right\} \end{aligned}$$

Now we will consider the representation of G, R, with linear transformations A, B, C

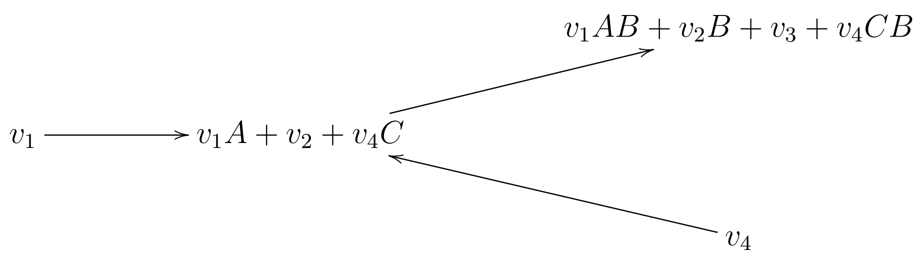
G



Each vector $v_i \in V_i$ is sent along all paths starting at i. Thus for v_1, we have the graph



We will then add up all the results for each vector at the vertices. Resulting in



So we send

$$[v_1, v_2, v_3, v_4] \mapsto [v_1, v_1A + v_2 + v_4C, v_1AB + v_2B + v_3 + v_4CB, v_4]$$

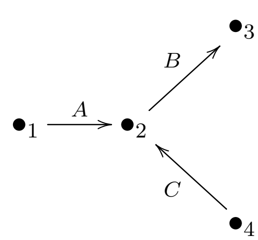
Using the matrix

$$\begin{bmatrix} F & F & F & 0 \\ 0 & F & F & 0 \\ 0 & 0 & F & 0 \\ 0 & F & F & F \end{bmatrix}$$

For example, starting with the element 1 from each F, this will give us

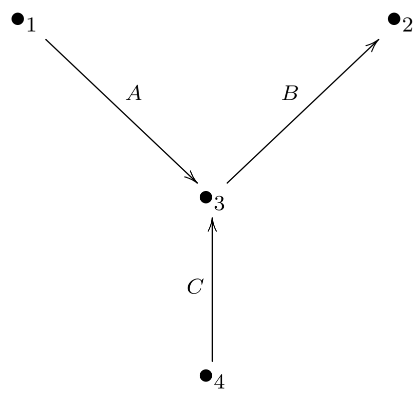
$$[v_1, v_2, v_3, v_4] \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = [v_1 \cdot 1, v_1 \cdot 1 + v_2 \cdot 1 + v_4 \cdot 1, v_1 \cdot 1 + v_2 \cdot 1 + v_3 \cdot 1 + v_4 \cdot 1, v_4 \cdot 1]$$

So for each $v_i \cdot 1$, 1 is the (i, j)-entry obtained by taking the appropriate linear transformations along the unique path from i to j. Thus



would result in $[v_1, v_1A + v_2 + v_4C, v_1AB + v_2B + v_3 + v_4CB, v_4]$.

Similarly for G



we would get $[v_1, v_1AB + v_2 + v_3B + v_4CB, v_1A + v_3 + v_4C, v_4]$ and the ring $\begin{bmatrix} F & F & F & 0 \\ 0 & F & 0 & 0 \\ 0 & F & F & 0 \\ 0 & F & F & F \end{bmatrix}$. Changing the 1's to arbitrary scalars would just scale each v_i. For example

$$[v_1, v_2, v_3, v_4] \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 3 & 1 & 2 \end{bmatrix} = [2v_1, 3v_1AB{+}4v_2{-}v_3B{+}3v_4CB, v_1A{-}v_3B{+}v_4C, 2v_4]$$

All of this implies that each representation of G is just a module over $End(\mathscr{P})$. As constructed, these would create right modules; however, using the same process one could construct these as left modules. We will change whenever it is convenient.

Example 5.7 From Lie Theory

A_2 $\quad \bullet \longrightarrow \bullet$ □

The ring, R, for $\bullet_1 \longrightarrow \bullet_2$ would come from

$$\begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$$

and would be

$$R = \left\{ \begin{bmatrix} a_1 & a_3 \\ 0 & a_2 \end{bmatrix} | a_i \in F \ all \ i \right\}$$

with some generic representation of $V_1 \xrightarrow{A} V_2$ giving a corresponding module of $V_1 \oplus V_2$ with the right action of R given by

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} = \begin{bmatrix} v_1 a_1 & v_1 a_3 A + v_2 a_2 \end{bmatrix}$$

Now consider A_3 $\bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3$.

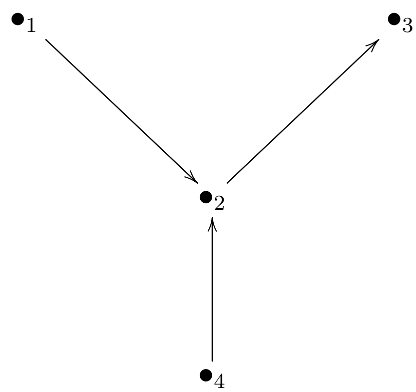
We have

$$R = \left\{ \begin{bmatrix} a_1 & a_4 & a_6 \\ 0 & a_2 & a_5 \\ 0 & 0 & a_3 \end{bmatrix} | a_i \in F \ all \ i \right\}$$

with a generic representation of $V_1 \xrightarrow{A} V_2 \xrightarrow{B} V_3$ making the corresponding module $V_1 \oplus V_2 \oplus V_3$ with right action given by

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} a_1 & a_4 & a_6 \\ 0 & a_2 & a_5 \\ 0 & 0 & a_3 \end{bmatrix} = \begin{bmatrix} v_1 a_1 & v_1 a_4 A + v_2 a_2 & v_1 a_6 AB + v_2 a_5 A + v_3 a_3 \end{bmatrix}$$

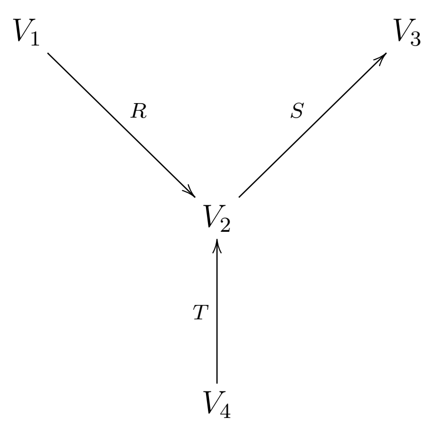
Consider G



Here $End(\mathscr{P}(1) \oplus \mathscr{P}(2) \oplus \mathscr{P}(3) \oplus \mathscr{P}(4))$ would be represented by the matrix

$$\begin{bmatrix} F & F & F & 0 \\ 0 & F & F & 0 \\ 0 & 0 & F & 0 \\ 0 & F & F & F \end{bmatrix}$$

and a generic representation looks like



Giving us a module $V_1 \oplus V_2 \oplus V_3 \oplus V_4$ over $End(\mathscr{P}(1) \oplus \mathscr{P}(2) \oplus \mathscr{P}(3) \oplus \mathscr{P}(4))$ with the right action defined by

$$\begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 & 0 \\ 0 & a_2 & b_2 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & a_4 & b_4 & c_4 \end{bmatrix}$$

$$= [\ v_1a_1 \quad v_1Rb_1 + v_2a_2 + v_4Ta_4 \quad v_1RSc_3 + v_2Sb_2 + v_3a_3 + v_4TSb_4 \quad v_4c_4\ ]$$

From this we will construct the following matrices.

$$e_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$e_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad F_{1,2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad F_{1,3} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_{2,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad F_{4,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad F_{4,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

In general for some edge $\bullet_i \longrightarrow \bullet_j$, e_i is the matrix with 1 in entry (i, i) and zero elsewhere. The matrix $F_{i,j}$ is the matrix with 1 in entry (i, j), zero elsewhere. Furthermore, these matrices satisfy the following properties.

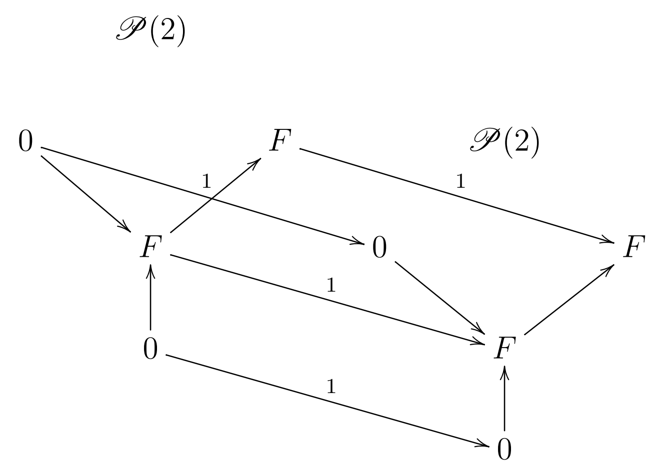
(i) $e_ie_j = \begin{cases} e_i & i = j \\ 0 & i \neq j \end{cases}$

(ii) $F_{i,j}e_k = \begin{cases} F_{i,j} & j = k \\ 0 & j \neq k \end{cases}$

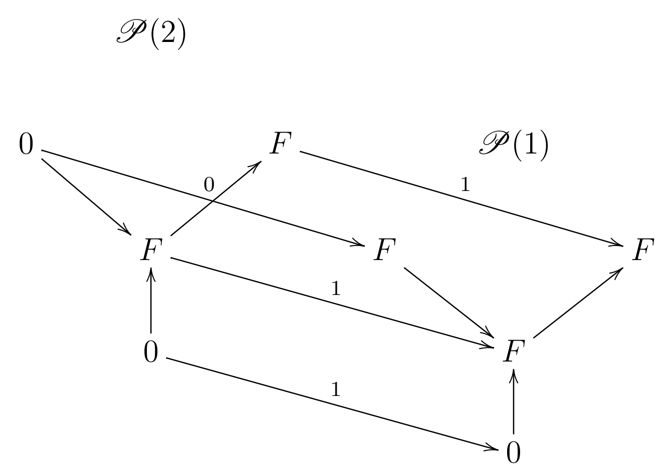
(iii) $e_kF_{i,j} = \begin{cases} F_{i,j} & k = i \\ 0 & k \neq i \end{cases}$

Since $End(\mathscr{P}) \cong [Hom(\mathscr{P}(i), \mathscr{P}(j))]$ and as previously stated these are just matrices made of elements from each F, e_i and $F_{i,j}$ represent endomorphisms. Each e_i will act as the identity on $\mathscr{P}(i)$ and send everything else to zero.

Continuing with the previous example, e_2 would be



Where all others would be mapped to zero. For $F_{i,j}$, the endomorphism will send all $\mathscr{P}(k)$ for $k \neq j$ and will send $\mathscr{P}(j)$ to $\mathscr{P}(i)$. For example,



We can make a connection between these matrices and the standard basis vectors of the vector space $M_{n\times n}(F)$. Using the notation $E^{(r,s)}$ we get $E^{(r,s)} = [a_{i,j}]$ where the entry $a_{i,j}$ is one if $(i,j) = (r,s)$ and zero if $(i,j) \neq (r,s)$.

Resulting in

$$E^{(r,s)}E^{(p,q)} = \begin{cases} 0 & \text{if } p \neq s \\ E^{(r,q)} & \text{if } p = s \end{cases}$$

Thus $e_i = E^{(i,i)}$ and $F_{i,j} = E^{(i,j)}$.

Given a module M over $End(\mathscr{P})$ each e_iM will be a vector space. Let $\widehat{M}$ be the representation which has e_iM on vertex i. If $\bullet_j \longrightarrow \bullet_i$ we can define a linear transformation

$$e_jM \xrightarrow{\widehat{F_{i,j}}} e_iM$$

by $e_jx \mapsto F_{i,j}(e_jx) = F_{i,j}x = e_iF_{i,j}x$.

This would then create a representation from a left $End(\mathscr{P})$ module. Thus giving a correspondence from $End(\mathscr{P}) - \mathbf{Mod}$ in $\mathbf{Rep}\, G$ as well as the correspondence from $\mathbf{Rep}\, G$ into $End(\mathscr{P}) - \mathbf{Mod}$. We would like these to be functors, so we must define their behaviour on morphisms.

Suppose $(V_i) \xrightarrow{\phi} (W_i)$ is a morphism of representations with corresponding modules $\oplus V_i$ and $\oplus W_i$. So for $(v_i) \in \oplus V_i$ we can define $\widetilde{\phi} : \oplus V_i \longrightarrow \oplus W_i$ by $\widetilde{\phi}(v_i) = (\phi_i v_i)$. This would be an additive function so it remains to check that $\widetilde{\phi}\lambda = \lambda\widetilde{\phi}$ for $\lambda \in End(\mathscr{P})$.

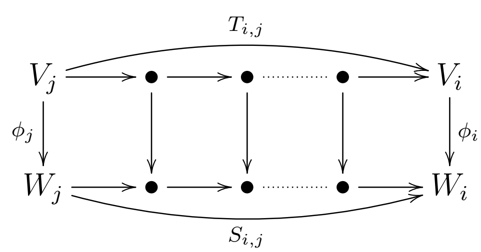
Each λ is represented as a matrix $[\lambda_{i,j}]$ where $\lambda_{i,j} \in F$ and $\lambda_{i,j} = 0$ if there exists no path from vertex j to vertex i. Let $T_{i,j}$ denote the unique path of linear transformations that represent that path, if no such path exists then $T_{i,j} = 0$. This results in

$$\lambda_{i,j}v_j = T_{i,j}v_j.$$

Now examining ϕ for n vertices, we have

$$\begin{aligned}\widetilde{\phi}(\lambda(v_j)) &= \widetilde{\phi}\left(\sum_{j=1}^{n}\lambda_{i,j}v_j\right)\\ &= \sum_{j=1}^{n}\widetilde{\phi}\lambda_{i,j}v_j\\ &= \sum_{j}\widetilde{\phi}T_{i,j}v_j\\ &= \phi_i T_{i,j}v_j\end{aligned}$$

Since ϕ is a morphism of representations we can create the following diagram.



Where $S_{i,j}$ is representation of the unique path of linear transformations between W_j and W_i with each square commuting. Thus $\phi_i T_{i,j} = S_{i,j}\phi_j$. Changing our sum to

$$\begin{aligned} &= \sum_{j} S_{i,j}\phi_j v_j\\ &= \sum_{j}\lambda_{i,j}\phi_j\\ &= \lambda(\widetilde{\phi}(v_j))\end{aligned}$$

Resulting in $\widetilde{\phi}$ a morphism of modules.

Conversely, if ψ is a morphism of modules from M to N. Then for each i, we get $\psi(e_i x) = e_i\psi(x)$ for all $x \in M$. Hence ψ is restricted to an additive map $\psi_i : e_iM \longrightarrow e_iN$. Also, if we regard e_iM and e_iN as vectors spaces, we get ψ as a linear transformation.

For any $a \in F$, we have

$$\begin{aligned}
\psi_i(ae_ix) &= \psi(ae_ix) && (5.1)\\
&= \psi(e_iax) && (5.2)\\
&= e_ia\psi(x) && (5.3)\\
&= a(e_i\psi(x)) && (5.4)
\end{aligned}$$

Where line (5.3) is justified by the fact that ψ is a module morphism. Line (5.2) relies on the fact $e_ia = ae_i$. However, this is easily seen from the definition of e_i as the matrix with $1 \in F$ in the (i,i)-entry. Since $1a = a1$ in F, we get $e_ia = ae_i$. Now for the edge $\bullet_j \longrightarrow \bullet_i$, we need to verify the following square commutes.

$$\begin{CD}
e_jM @>{\widehat{F_{i,j}}}>> e_iM\\
@V{\psi_j}VV @VV{\psi_i}V\\
e_jN @>>{\widehat{F_{i,j}}}> e_iN
\end{CD}$$

For each $x \in M$ we have

$$\begin{aligned}
\psi\widehat{F_{i,j}}(e_jx) &= \psi_i(F_{i,j}e_jx) && \text{by definition of } \widehat{F_{i,j}}\\
&= \psi_i(e_iF_{i,j}x) && \text{by properties of } e_i \text{ and } F_{i,j}\\
&= \psi(e_iF_{i,j}x) && \text{by definition of } \psi\\
&= F_{i,j}\psi(e_jx) && \psi \text{ a module morphism and } F_{i,j} \in End(\mathscr{P})\\
&= \widehat{F_{i,j}}\psi_j(e_jx) && \text{definition of } \widehat{F_{i,j}} \text{ and } \psi
\end{aligned}$$

Thus the square commutes resulting in $\psi \longrightarrow (\psi_i)$ sends a morphism of modules into a morphism of representations. Giving two functors F_1 and F_2 such that

$$\mathbf{Rep}\,G \underset{F_2}{\overset{F_1}{\rightleftarrows}} End(\mathscr{P}) - \mathbf{Mod}$$

Given a representation R of G we claim $F_2F_1(R) \cong R$ and if $R \xrightarrow{\phi} S$ the following square commutes.

$$\begin{array}{ccc} F_2F_1(R) & \longrightarrow & R \\ {\scriptstyle F_2F_1(\phi)} \downarrow & & \downarrow {\scriptstyle \phi} \\ F_2F_1(S) & \longrightarrow & S \end{array}$$

It can be shown from the construction of F_1 and F_2. Similarly, if M is a module we get $F_1F_2(M) \cong M$ and the commutative square,

$$\begin{array}{ccc} F_1F_2(M) & \longrightarrow & M \\ {\scriptstyle F_1F_2(\psi)} \downarrow & & \downarrow {\scriptstyle \psi} \\ F_1F_2(N) & \longrightarrow & N \end{array}$$

for a module morphism $\psi : M \longrightarrow N$.

5.2 Analysis of objects in RepG

Taking a closer look at a typical object in **Rep** $(\bullet \longrightarrow \bullet)$. So given $V \xrightarrow{T} W$ we can decompose the vector spaces V and W as follows:

$$\begin{aligned} V = & \quad \ker T \oplus V' \quad \text{for some } V' \\ W = & \quad \operatorname{Im} T \oplus W' \quad \text{for some } W' \end{aligned}$$

as internal direct sums. In other words, for $v \in V$ we get $v = x + y$ where $x \in \ker T$ and $y \in V'$ with $\ker T \cap V' = \emptyset$ and $v = 0$ if and only if $x = y = 0$. We can thus decompose this as

$$V' \oplus \ker T \xrightarrow{\begin{bmatrix} T' & 0 \\ 0 & 0 \end{bmatrix}} \operatorname{Im} T \oplus W'$$

where $T' = T|_{V'}$. Resulting in this being isomorphic to $V \xrightarrow{T} W$, now as external sums. For $v \in V$ we have $v = v' + k$, and $v \mapsto (v', k)$ for $v' \in V'$ and $k \in \ker T$ as well as $w = Tu + w'$ for some $u \in V$. Thus $w \mapsto (Tu, w')$. Giving the following diagram,

$$\begin{array}{ccc} V & \xrightarrow{\quad T \quad} & W \\ \cong \big\downarrow & & \big\downarrow \cong \\ V' \oplus \ker T & \xrightarrow[\begin{bmatrix} T' & 0 \\ 0 & 0 \end{bmatrix}]{} & \operatorname{Im} T \oplus W' \end{array}$$

where

$$\begin{array}{ccc} v & \longrightarrow & Tv \\ \big\downarrow & & \big\downarrow \\ (v', k) & \xrightarrow[\begin{bmatrix} T' & 0 \\ 0 & 0 \end{bmatrix}]{} & (Tv, 0) \end{array}$$

which is commutative since

$$\begin{aligned} T(v) &= T(v' + k) \\ &= T(v') + T(k) \\ &= Tv \end{aligned}$$

We can express two representations as a direct sum as follows. Given two representations R and S define the direct sum to be the direct sum of vector spaces at each vertex and $R \oplus S : V_1 \oplus W_1 \longrightarrow V_2 \oplus W_2$ to be a linear transformation $R \oplus S(v, w) = (Rv, Sw)$ for all $v \in V_1$ and $w \in W_1$. So the sum of

$$V_1 \xrightarrow{\quad R \quad} V_2 \qquad \oplus \qquad W_1 \xrightarrow{\quad S \quad} W_2$$

would be

$$V_1 \oplus W_1 \xrightarrow{R \oplus S} V_2 \oplus W_2$$

We can express

$$V' \oplus \ker T \xrightarrow{\begin{bmatrix} T' & 0 \\ 0 & 0 \end{bmatrix}} \operatorname{Im} T \oplus W'$$

as the direct sum of two representations as follows.

$$V' \xrightarrow{T'} \operatorname{Im} T$$

$$\oplus$$

$$\ker T \xrightarrow{0} W'$$

Generally speaking, $A \xrightarrow{0} B$ is the direct sum of

$$A \longrightarrow 0$$

$$\oplus$$

$$0 \longrightarrow B$$

Thus

$$V \xrightarrow{T} W \cong (V' \xrightarrow{T'} \operatorname{Im} T) \oplus (\ker T' \xrightarrow{0} W')$$

$$\cong (V' \xrightarrow{T'} \operatorname{Im} T) \oplus (\ker T \longrightarrow 0) \oplus (0 \longrightarrow W')$$

However, T' is an isomorphism of vector spaces, thus there is an isomorphism of representations

$$\begin{array}{ccc} V' & \xrightarrow{T'} & \operatorname{Im} T \\ \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle T'^{-1}} \\ V' & = & V' \end{array}$$

Where I is the identity linear transformation. Giving $V' \xrightarrow{T'} \operatorname{Im} T \cong V' \xrightarrow{I} V'$. Now if $V = (\oplus_B F)$ where B is the basis, then $V' \xrightarrow{I} V' \cong \oplus(F \xrightarrow{1} F)$. Thus we get $V' \longrightarrow V'$ is projective.

Similarly, for zero morphisms

$$0 \longrightarrow W$$

$$W \longrightarrow 0$$

we get these as direct sums of $0 \longrightarrow F$ and $F \longrightarrow 0$ respectively. Hence any representation of $\bullet \longrightarrow \bullet$ is a direct sum of three types of representations.

$$F \xrightarrow{1} F \qquad 0 \xrightarrow{0} F \qquad F \xrightarrow{0} 0$$

Where

- $F \xrightarrow{1} F$ and $0 \xrightarrow{0} F$ are projective;
- $F \xrightarrow{1} F$ and $F \xrightarrow{0} 0$ are injective;
- $F \xrightarrow{0} 0$ and $0 \xrightarrow{0} F$ are simple

with simple meaning that there are no non-trivial proper subobjects.

A typical projective element in **Rep** $(\bullet \longrightarrow \bullet)$ will be the sum of the two projective types and will have the form (up to isomorphism)

$$V \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} V \oplus W$$

$$v \longmapsto (v, 0)$$

If T is one-to-one, then $\ker T = 0$. Giving $V \xrightarrow{T} W \cong V' \xrightarrow{T'} \operatorname{Im} T \oplus 0 \longrightarrow W'$ and thus projective. Likewise, if T is onto is, then $\operatorname{Im} T = W$ and hence

$W' = 0$. Giving $V \xrightarrow{T} W \cong V' \xrightarrow{T'} \operatorname{Im} T \oplus \ker T \longrightarrow 0$ and thus injective. However, any one-to-one linear transformation $V \xrightarrow[1-1]{T} Z$ can be decomposed into

$$V \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} V \oplus W$$

where $W \oplus \operatorname{Im} T = Z$. Which gives the following diaram.

$$\begin{array}{ccc} V & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & V \oplus W \\ \Vert & & \downarrow \begin{bmatrix} T & 1 \end{bmatrix} \\ V & \xrightarrow[T]{} & Z \end{array}$$

Where the top row gives

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} v = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

and the right-hand column gives

$$\begin{bmatrix} T & 1 \end{bmatrix} \begin{bmatrix} v \\ 0 \end{bmatrix} = Tv + w$$

Which results in an isomorphism. Meaning that all projectives have, up to isomorphism, the form $V \xrightarrow{T} W$ where T is one-to-one.

The dual notion for injectives is as follows. All injective elements will have the form, up to isomorphism:

$$V \oplus W \xrightarrow{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} V$$

$$(v, 0) \longmapsto v$$

resulting in the following diagram.

$$\begin{array}{ccc} V \oplus W & \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} & V \\ \begin{bmatrix}S\\1\end{bmatrix}\Big\downarrow & & \Big\| \\ Z & \xrightarrow[S]{\hspace{2em}} & V \end{array}$$

Where S is an onto function. Thus injectives will have, up to isomorphism, the form $V \xrightarrow[S]{} W$ with S onto.

Now looking at projective presentations of $V \xrightarrow{T} W$. Since we can decompose $V \xrightarrow{T} W$ into three types, we just need to consider the non-projective case of $\ker T \longrightarrow 0$. Generally speaking, if $U \longrightarrow 0$ we can create

$$\begin{array}{ccc} U & \xrightarrow{1} & U \\ \Big\| & & \Big\downarrow \\ U & \longrightarrow & 0 \end{array}$$

To find a projective presentation for $V \xrightarrow{T} W$ we will use this diagram and add on $\ker T \xrightarrow{1} \ker T$, creating

$$\begin{array}{ccc} V' \oplus \ker T & \xrightarrow{\begin{bmatrix}T & 0\\0 & 0\\0 & 1\end{bmatrix}} & \operatorname{Im} T \oplus W' \oplus \ker T \\ \Big\downarrow 1 & & \Big\downarrow \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix} \\ V' \oplus \ker T & \xrightarrow{\begin{bmatrix}T' & 0\\0 & 0\end{bmatrix}} & \operatorname{Im} T \oplus W' \end{array}$$

The kernel of this presentation is $0 \xrightarrow{0} \ker T$ which is also projective. This leads us to the following.

Theorem 5.8 For projective objects in **Rep** $(\bullet \longrightarrow \bullet)$, subobjects are also projective.

Proof: Suppose $A \xrightarrow{S} B$ is projective. Thus S is a one-to-one transformation. Now for a monomorphism to $A \xrightarrow{S} B$ we can create the following diagram,

$$\begin{array}{ccc} A' & \xrightarrow{S'} & B' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[S]{} & B' \end{array}$$

where f and g are one-to-one linear transformations. Since the composition of two one-to-one functions is also one-to-one, $f \circ g$ would then be one-to-one. This would imply that $g \circ S'$ is one-to-one. Hence $A' \xrightarrow{S'} B'$ is projective. ∎

<u>**Note:**</u> The dual notion for injectives would use quotients. In other words, the quotients of injectives are injectives.

A projective presentation for a general tree G would have each edge, $V_i \xrightarrow{T_{i,j}} V_j$, one-to-one. Similarly, for an injective presentation each $T_{i,j}$ would need to onto.

By reducing the problem to each edge, we can create the following squares.

$$\begin{array}{ccc} V_i' & \xrightarrow{T_{i,j}'} & V_j' \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_j} \\ V_i & \xrightarrow[T_{i,j}]{} & V_j \end{array}$$

Now for a typical object in **Rep** $(\bullet \longrightarrow \bullet \longrightarrow \bullet)$. We have representations of the form $U \xrightarrow{T} V \xrightarrow{S} W$. We can express U as $U = T^{-1}(\ker S) \oplus X$ for some vector space X, where T^{-1} is the pre-image under T. Giving $\ker T \subseteq T^{-1}(\ker S)$. So $T^{-1}(\ker S) = Y \oplus \ker T$ for some Y. Now $U = X \oplus Y \oplus \ker T$.

Which implies that T is one-to-one on $X \oplus Y$ and

$$\operatorname{Im} T = T(X \oplus Y) = T(X) \oplus T(Y)$$

Now $T(Y) \subseteq \ker S$, so $\ker S = T(Y) \oplus K$ for some K, and since $U = T^{-1}(\ker S) \oplus X$, $T(X) \cap \ker S = \emptyset$. Thus we can express V as

$$V = T(X) \oplus T(Y) \oplus K \oplus Z$$

for some Z. Since $T(Y) \oplus K = \ker S$, S will be one-to-one on $T(X) \oplus Z$, giving $\operatorname{Im} S = ST(X) \oplus S(Z)$. Now consider $u \in U$ with $u = x + y + k$ where $x \in X,\ y \in Y\ k \in \ker T$. Then

$$\begin{aligned} STu = &\quad ST(x+y+k) \\ = &\quad STx + STy + STk \\ = &\quad STx + STy \qquad STk = 0 \\ = &\quad STx \qquad Ty \in \ker S \end{aligned}$$

Thus $\operatorname{Im} ST = ST(X)$. Finally, $W = ST(X) \oplus S(Z) \oplus L$ for some L. Allowing $U \xrightarrow{T} V \xrightarrow{W}$ to be expressed as

$$X \oplus Y \oplus \ker T \xrightarrow{\alpha} T(X) \oplus T(Y) \oplus K \oplus Z \xrightarrow{\beta} ST(X) \oplus L \oplus S(Z)$$

where

$$\alpha = \begin{bmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \beta = \begin{bmatrix} S & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S \end{bmatrix}$$

Giving $U \xrightarrow{T} V \xrightarrow{S} W$ "breakable" into

$$X \longrightarrow TX \longrightarrow STX$$

$$Y \longrightarrow TY \longrightarrow 0$$

$$\ker T \longrightarrow 0 \longrightarrow 0$$

$$0 \longrightarrow K \longrightarrow 0$$

$$0 \longrightarrow Z \longrightarrow SZ$$

$$0 \longrightarrow 0 \longrightarrow L$$

Implying that for any representation $U \longrightarrow V \longrightarrow W$ can be expressed as the direct sum of the following six indecomposable objects.

- $F \longrightarrow F \longrightarrow F$, injective and projective;
- $F \longrightarrow F \longrightarrow 0$, injective;
- $F \longrightarrow 0 \longrightarrow 0$, injective and simple;
- $0 \longrightarrow F \longrightarrow 0$, simple;
- $0 \longrightarrow F \longrightarrow F$, projective;
- $0 \longrightarrow 0 \longrightarrow F$, projective and simple.

5.3 Computing Ext

For Ext in $\mathbf{Rep}\, G$ we must first know what an exact sequence looks like. Given G,$\bullet \longrightarrow \bullet$ and exact sequence is of the form

$$\begin{array}{ccc} U_1 & \xrightarrow{R} & U_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle g_1} \\ V_1 & \xrightarrow{S} & V_2 \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle g_2} \\ W_1 & \xrightarrow{T} & W_2 \end{array}$$

Here the columns, $U_1 \xrightarrow{f_1} V_1 \xrightarrow{f_2} W_1$, are the exact sequences; as vector spaces, this would force $V_i \cong U_i \oplus W_i$.

Given $V_i \xrightarrow{T_i} W_i$ for $i = 1, 2$ we want to compute $Ext(V_1 \xrightarrow{T_1} W_1, V_2 \xrightarrow{T_2} W_2)$. Since $V_1' \longrightarrow \operatorname{Im} T_1$ and $0 \longrightarrow W_1'$ are both projective and $V_2' \longrightarrow \operatorname{Im} T_2$ and $\ker T_2 \longrightarrow 0$ are both injective, we find that $Ext(V_1 \xrightarrow{T_1} W_1, V_2 \xrightarrow{T_2} W_2) \cong Ext(\ker T_1 \longrightarrow 0, 0 \longrightarrow W_2')$. Which reduces the problem to finding the extensions of the form $Ext(V \longrightarrow 0, 0 \longrightarrow W)$. Elements of this would be extensions of the form

$$\begin{array}{ccccc} 0 & \longrightarrow & E_1 & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ W & \longrightarrow & E_2 & \longrightarrow & 0 \end{array}$$

For any $E_1 \longrightarrow V$ and $W \longrightarrow E_2$ this diagram is commutative. However, for the rows to be exact, $E_1 \longrightarrow V$ and $W \longrightarrow E_2$ must be isomorphisms. This implies that $Ext(V \longrightarrow 0, 0 \longrightarrow W) \cong Hom(V, W)$. (Where $Hom(V, W)$ is taken in the category of vector spaces.) So in general

$$\begin{aligned} Ext(V_1 \xrightarrow{T_1} W_1, V_2 \xrightarrow{T_2} W_2) &\cong Ext(\ker T_1 \longrightarrow 0, 0 \longrightarrow W_2) \\ &\cong Hom(\ker T_1, W_2') \end{aligned}$$

Where $W_2' \oplus \operatorname{Im} T_2 = W_2$, so set $W_2' = (\operatorname{Im} T_2)^C$.

Now a representation of the form $\bullet \longrightarrow \bullet \longrightarrow \bullet$ has extensions of the form:

$$\begin{array}{ccccc} U_1 & \xrightarrow{R_1} & U_2 & \xrightarrow{R_2} & U_3 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle h_1} \\ V_1 & \xrightarrow{S_1} & V_2 & \xrightarrow{S_2} & V_3 \\ \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle h_2} \\ W_1 & \xrightarrow{T_1} & W_2 & \xrightarrow{T_2} & W_3 \end{array}$$

Where each column is exact.

For finite sums $Ext(\oplus_i A_i, \oplus_j B_j) \cong \oplus_{i,j} Ext(A_i, B_j)$, so we can just consider the six basic forms of $\bullet \longrightarrow \bullet \longrightarrow \bullet$. Our problem is further reduced by the fact that $Ext(A, -) = 0$ and $Ext(-, B) = 0$ where A is projective and B is injective. Giving only nine possible scenarios.

$$\bullet Ext(A \longrightarrow 0 \longrightarrow 0, 0 \longrightarrow 0 \longrightarrow B)$$

This would create the extension

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \beta \\ E_1 & \longrightarrow & E_2 & \longrightarrow & E_3 \\ \downarrow \alpha & & \downarrow & & \downarrow \\ A & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

For this to be exact, $E_2 = 0$ with α and β would have to be isomorphisms. Thus the middle term would be of the form

$$A \longrightarrow 0 \longrightarrow B$$

Which splits and hence decomposes into

$$A \longrightarrow 0 \longrightarrow 0 \ \oplus \ 0 \longrightarrow 0 \longrightarrow B$$

and the extension forms a split exact sequence. Thus $Ext(A \longrightarrow 0 \longrightarrow 0, 0 \longrightarrow 0 \longrightarrow B) = 0$.

$$\bullet Ext(A \longrightarrow 0 \longrightarrow 0, 0 \longrightarrow B \longrightarrow 0)$$

Here extensions look like

$$\begin{array}{ccccc} 0 & \longrightarrow & B & \longrightarrow & 0 \\ \downarrow & & \downarrow \beta & & \downarrow \\ E_1 & \xrightarrow{R} & E_2 & \longrightarrow & E_3 \\ \downarrow \alpha & & \downarrow & & \downarrow \\ A & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Which would make $E_3 = 0$ with α and β isomorphisms. Here, the middle term would be of the form

$$A \xrightarrow{R} B \longrightarrow 0$$

for some R. The resulting short exact sequence does not split and it follows that $Ext(A \longrightarrow 0 \longrightarrow 0, 0 \longrightarrow B \longrightarrow 0) \cong Hom(A, B)$ where $Hom(A, B)$ is taken in the category of vector spaces.

$$\bullet Ext(A \longrightarrow 0 \longrightarrow 0, 0 \longrightarrow B \xrightarrow{1} B)$$

Here extension have the form

$$\begin{array}{ccccc}
0 & \longrightarrow & B & \xrightarrow{1} & B \\
\downarrow & & \downarrow{\scriptstyle \beta_1} & \Sigma & \downarrow{\scriptstyle \beta_2} \\
E_1 & \xrightarrow{R_1} & E_2 & \xrightarrow{R_2} & E_3 \\
\downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow \\
A & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Here exactness gives us that α, β_1, and β_2 are all isomorphisms. Commutativity of the square Σ forces $\beta_2 = R_2\beta_1$, which implies that $R_2 = \beta_2\beta_1^{-1}$ is also an isomorphism. Thus the above is equivalent to

$$\begin{array}{ccccc}
A & \xrightarrow{\beta^{-1}R_1\alpha^{-1}} & B & \xrightarrow{1} & B \\
\downarrow{\scriptstyle \alpha^{-1}} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_2} \\
E_1 & \xrightarrow{R_1} & E_2 & \xrightarrow{R_2} & E_3
\end{array}$$

It follows that the middle term will have the form

$$A \xrightarrow{R_1} B \xrightarrow{1} B$$

for any linear transformation R_1. Hence $Ext(A \longrightarrow 0 \longrightarrow 0, 0 \longrightarrow B \xrightarrow{1} B) \cong Hom(A, B)$.

$$\bullet Ext(A \xrightarrow{1} A \longrightarrow 0, 0 \longrightarrow 0 \longrightarrow B)$$

This gives the extensions the form

$$\begin{CD}
0 @>>> 0 @>>> B \\
@VVV @VVV @VV{\beta}V \\
E_1 @>{R_1}>> E_2 @>{R_2}>> E_3 \\
@V{\alpha_1}V{\Sigma}V @VV{\alpha_2}V @VVV \\
A @>{1}>> A @>>> 0
\end{CD}$$

Much like the previous example, α_1, α_2, and β are all isomorphisms as well square Σ giving $R_1 = \alpha_2^{-1}\alpha_1$. Making all the middle terms have the form

$$A \xrightarrow{1} A \xrightarrow{R_2} B$$

It follows $Ext(A \xrightarrow{1} A \longrightarrow 0, 0 \longrightarrow 0 \longrightarrow B) \cong Hom(A, B)$.

$$\bullet Ext(A \xrightarrow{1} A \longrightarrow 0, 0 \longrightarrow B \longrightarrow 0)$$

We get the diagram

$$\begin{CD}
0 @>>> B @>>> 0 \\
@VVV @VV{\beta}V @VVV \\
E_1 @>{R_1}>> E_2 @>{R_2}>> E_3 \\
@V{\alpha_1}V{\Sigma}V @VV{\alpha_2}V @VVV \\
A @>{1}>> A @>>> 0
\end{CD}$$

Here the column exactness would force $E_3 = 0$, β to be one-to-one, α_1 to be an isomorphism, α_2 to be onto and $\operatorname{Im}\beta = \ker\alpha_2$. From the commutativity of square Σ we get $\alpha_2 R_1 = R_2$, which implies that R_1 is one-to-one as well. Since α_1 is one-to-one and $\alpha_1 = \alpha_2 R_1$, then

$$\begin{aligned}
& \operatorname{Im} R_1 \cap \ker\alpha_2 &= 0 \\
\Rightarrow \quad & \operatorname{Im} R_1 \cap \operatorname{Im}\beta &= 0
\end{aligned}$$

Now given $y \in E_2$, $\alpha_2 y \in A$ and α_1 and isomorphism, we have

$$\alpha_2 y = \alpha_1 x \qquad \text{some } x \in E_1$$

$$\alpha_2 y = \alpha_2 R_1 x$$

$$\alpha_2 (y - R_1 x) = 0$$

$$\Rightarrow \quad (y - R_1 x) \in \ker \alpha_2 \;\; = \operatorname{Im} \beta$$

$$\Rightarrow \quad y \in \operatorname{Im} R_1 + \operatorname{Im} \beta$$

This would imply that $E_2 = \operatorname{Im} R_1 \oplus \operatorname{Im} \beta$. Giving

$$\begin{array}{ccccc} A & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & A \oplus B & \longrightarrow & 0 \\ \downarrow{\scriptstyle \alpha_1^{-1}} & & \downarrow{\scriptstyle \begin{bmatrix} R_1 \alpha_1^{-1} & \beta \end{bmatrix}} & & \| \\ E_1 & \longrightarrow & E_2 & \longrightarrow & 0 \end{array}$$

as an isomorphism. Here the middle terms have the form

$$A \longrightarrow A \oplus B \longrightarrow 0$$

which decompose to $A \longrightarrow A \longrightarrow 0 \oplus 0 \longrightarrow B \longrightarrow 0$, and the resulting exact sequence splits. Hence

$Ext(A \longrightarrow A \longrightarrow 0, 0 \longrightarrow B \longrightarrow 0) = 0$

$$\bullet Ext(A \xrightarrow{1} A \longrightarrow 0, 0 \longrightarrow B \xrightarrow{1} B)$$

This give us the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & B & \xrightarrow{1} & B \\ \downarrow & & \downarrow{\scriptstyle \beta_1} \;\; \Sigma_1 & & \downarrow{\scriptstyle \beta_2} \\ E_1 & \xrightarrow{R_1} & E_2 & \xrightarrow{R_2} & E_3 \\ \downarrow{\scriptstyle \alpha_1} \;\; \Sigma_2 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow \\ A & \xrightarrow{1} & A & \longrightarrow & 0 \end{array}$$

Here exactness would force α_1 and β_2 to be isomorphisms, β_1 to be one-to-one, α_2 to be onto, and $\operatorname{Im}\beta_1 = \ker\alpha_2$. From the commutativity of square Σ_1 we find $R_2\beta_1 = \beta_2$ and thus R_2 is onto. From the commutativity of square Σ_2 we find $\alpha_2 R_1 = \alpha_1$ and thus R_1 is one-to-one. As before, since R_1 is one-to-one and $\operatorname{Im}\beta_1 = \operatorname{Im}\alpha_2$, we have $\operatorname{Im} R_1 \cap \operatorname{Im}\beta_1 = 0$.

Then given a $y \in E_2$

$$\begin{aligned}
\alpha_2 y &= \alpha_1 x \qquad \text{some } x \in E_1\\
\alpha_2 y &= \alpha_2 R_1 x\\
\alpha_2(y - R_1 x) &= 0\\
\Rightarrow \quad (y - R_1 x) &\in \ker\alpha_2 \;= \operatorname{Im}\beta\\
\Rightarrow \quad y &\in \operatorname{Im} R_1 + \operatorname{Im}\beta
\end{aligned}$$

We would get

$$\begin{array}{ccccc}
A & \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} & A\oplus B & \xrightarrow{\begin{bmatrix}0 & 1\end{bmatrix}} & B\\
\downarrow{\scriptstyle \alpha_1^{-1}} & & \downarrow{\scriptstyle \begin{bmatrix}R_1\alpha_1^{-1} & \beta_1\end{bmatrix}} & & \downarrow{\scriptstyle \beta_2}\\
E_1 & \xrightarrow{R_1} & E_2 & \xrightarrow{R_2} & E_3
\end{array}$$

as an isomorphism. Thus the middle terms have the form

$$A \longrightarrow A \oplus B \longrightarrow B$$

Which can be decomposed as $A \longrightarrow A \longrightarrow 0 \oplus 0 \longrightarrow B \longrightarrow B$, and the corresponding exact sequence splits, so

$Ext(A \longrightarrow A \longrightarrow 0, 0 \longrightarrow B \longrightarrow B) = 0.$

$$\bullet Ext(0 \longrightarrow A \longrightarrow 0, 0 \longrightarrow 0 \longrightarrow B)$$

This gives the following diagram.

$$\begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & B \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \beta} \\
E_1 & \xrightarrow{R_1} & E_2 & \xrightarrow{R_2} & E_3 \\
\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow \\
0 & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

Here exactness forces α and β to both be isomorphisms and $E_1 = 0$. Which gives

$$\begin{array}{ccccc}
0 & \longrightarrow & A & \longrightarrow & B \\
\downarrow & & \downarrow{\scriptstyle \alpha^{-1}} & & \downarrow{\scriptstyle \beta} \\
E_1 & \longrightarrow & E_2 & \longrightarrow & E_3
\end{array}$$

as an isomorphism. Thus the middle terms have the form

$$0 \longrightarrow A \longrightarrow B$$

and it follows that $Ext(0 \longrightarrow A \longrightarrow 0, 0 \longrightarrow 0 \longrightarrow B) \cong Hom(A, B)$.

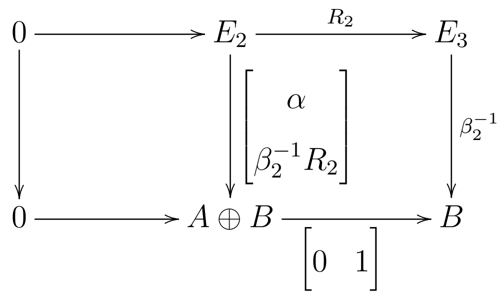
$$\bullet Ext(0 \longrightarrow A \longrightarrow 0, 0 \longrightarrow B \xrightarrow{1} B)$$

This will give the following diagram.

$$\begin{array}{ccccc}
0 & \longrightarrow & B & \xrightarrow{1} & B \\
\downarrow & & \downarrow{\scriptstyle \beta_1} & \Sigma & \downarrow{\scriptstyle \beta_2} \\
E_1 & \xrightarrow{R_1} & E_2 & \xrightarrow{R_2} & E_3 \\
\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow \\
0 & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

Here exactness will force $E_1 = 0$, β_2 to be an isomorphism, α to be onto, β_1 to be one-to-one, and $\operatorname{Im} \beta_1 = \ker \alpha$. The commutativity of square Σ

will give $R_2\beta_1 = \beta_2$ with R_2 onto. This yields the following diagram.



We would like this diagram to be an isomorphism. Given $(a, b) \in A \oplus B$, we have $a = \alpha y$ for some $y \in E_2$ since α is onto. Now consider $y + \beta_1(b - \beta_2^{-1}R_2(y))$. Since $\ker \alpha = \operatorname{Im} \beta_1$, we find

$$\begin{aligned} \alpha(y + \beta_1(b - \beta_2^{-1}R_2(y))) &= \alpha(y) + \alpha(\beta_1(b - \beta_2^{-1}R_2(y))) \\ &= \alpha(y) \\ &= a \end{aligned}$$

and

$$\begin{aligned} \beta_2^{-1}R_2(y + \beta_1(b - \beta_2^{-1}R_2(y))) &= \beta_2^{-1}R_2(y) + \beta_2^{-1}R_2\beta_1(b - \beta_2^{-1}R_2(y)) \\ &= \beta_2^{-1}R_2(y) + \beta_2^{-1}R_2\beta_1(b) - \beta_2^{-1}R_2(y) \\ &= \beta_2^{-1}R_2\beta_1(b) \\ &= \beta_2^{-1}\beta_2(b) \\ &= b \end{aligned}$$

So we get $\begin{bmatrix} \alpha \\ \beta_2^{-1}R_2 \end{bmatrix}$ as onto. If $\begin{bmatrix} \alpha(y) \\ \beta_2^{-1}R_2(y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then $y \in \ker \alpha = \operatorname{Im} \beta_1$ and $y = \beta_1(b)$ for some b. Giving

$$\begin{aligned} 0 &= \beta_2^{-1}R_2(y) \\ &= \beta_2^{-1}R_2\beta_1(b) \\ &= \beta_2^{-1}\beta_2(b) \\ &= b \end{aligned}$$

Thus $b = 0$. So $y = \beta_1(b) = \beta_1(0) = 0$. Meaning $\begin{bmatrix} \alpha \\ \beta_2^{-1} R_2 \end{bmatrix}$ is one-to-one and therefore an isomorphism. Therefore the middle terms will have the form

$$0 \longrightarrow A \oplus B \longrightarrow B$$

Which decomposes as $0 \longrightarrow A \longrightarrow 0 \oplus 0 \longrightarrow B \longrightarrow B$, which splits. Giving $Ext(0 \longrightarrow A \longrightarrow 0, 0 \longrightarrow B \longrightarrow B) = 0$.

$$\bullet Ext(0 \longrightarrow A \longrightarrow 0, 0 \longrightarrow B \longrightarrow 0)$$

This give the following diagram.

$$\begin{array}{ccccc}
0 & \longrightarrow & B & \longrightarrow & 0 \\
\downarrow & & \downarrow \beta & & \downarrow \\
E_1 & \xrightarrow{R_1} & E_2 & \xrightarrow{R_2} & E_3 \\
\downarrow & & \downarrow \alpha & & \downarrow \\
0 & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

Here exactness would force $E_1 = E_3 = 0$, α to be onto, β to be one-to-one, $\operatorname{Im} \beta = \ker \alpha$. We can create the following digram.

$$\begin{array}{ccccc}
0 & \longrightarrow & E_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow \begin{bmatrix} \alpha \\ \hat{\beta} \end{bmatrix} & & \downarrow \\
0 & \longrightarrow & A \oplus B & \longrightarrow & 0
\end{array}$$

Where $\hat{\beta}$ is obtained by splitting β. Then

$$B \underset{\curvearrowleft}{\longrightarrow} E_2 = \operatorname{Im} \beta \oplus Z \qquad \text{some } Z$$

In other words, each $y \in E_2$ can be expressed uniquely as $y = \beta(b) + z$ for some $z \in Z$ where $\hat{\beta}(y) = b$.

Here we get the middle terms having the form $0 \longrightarrow A \oplus B \longrightarrow 0$. Which can be decomposed as $0 \longrightarrow A \longrightarrow 0 \oplus 0 \longrightarrow B \longrightarrow 0$ and again the corresponding short exact sequence splits. Thus
$Ext(0 \longrightarrow A \longrightarrow 0, 0 \longrightarrow B \longrightarrow 0) = 0$.

To summarize these results we have

(i) $Ext(A \longrightarrow 0 \longrightarrow 0, 0 \longrightarrow 0 \longrightarrow B) = 0$

(ii) $Ext(A \longrightarrow 0 \longrightarrow 0, 0 \longrightarrow B \longrightarrow 0) \cong Hom(A, B)$

(iii) $Ext(A \longrightarrow 0 \longrightarrow 0, 0 \longrightarrow B \longrightarrow B) \cong Hom(A, B)$

(iv) $Ext(A \longrightarrow A \longrightarrow 0, 0 \longrightarrow 0 \longrightarrow B) \cong Hom(A, B)$

(v) $Ext(A \longrightarrow A \longrightarrow 0, 0 \longrightarrow B \longrightarrow 0) = 0$

(vi) $Ext(A \longrightarrow A \longrightarrow 0, 0 \longrightarrow B \longrightarrow B) = 0$

(vii) $Ext(0 \longrightarrow A \longrightarrow 0, 0 \longrightarrow 0 \longrightarrow B) \cong Hom(A, B)$

(viii) $Ext(0 \longrightarrow A \longrightarrow 0, 0 \longrightarrow B \longrightarrow B) = 0$

(ix) $Ext(0 \longrightarrow A \longrightarrow 0, 0 \longrightarrow B \longrightarrow 0) = 0$

More generally, given two representations, $U_i \xrightarrow{T_i} V_i \xrightarrow{S_i} W_i$, for $i = 1, 2$, we know we can decompose these into direct sums of the following.

(1) $X_i \longrightarrow T_i X_i \longrightarrow S_i T_i X_i$

(2) $Y_i \longrightarrow T_i Y_i \longrightarrow 0$

(3) $\ker T_i \longrightarrow 0 \longrightarrow 0$

(4) $0 \longrightarrow K_i \longrightarrow 0$

(5) $0 \longrightarrow Z_i \longrightarrow S_i Z_i$

(6) $0 \longrightarrow 0 \longrightarrow L_i$

Since $\operatorname{Im} T_i \oplus K_i \oplus Z_i = V_i$ and $\operatorname{Im} S_i \oplus L_i = W_i$, for clarity let $K_i = (\operatorname{Im} T_i)^C$ and $L_i = (\operatorname{Im} S_i)^C$. Omitting the times when $Ext(-,-) = 0$, we have the situations $(ii), (iii), (iv)$, and (vii) from the summary above to consider. Situation (ii) will use (3) and (4) and give $Ext(\ker T_i \longrightarrow 0 \longrightarrow 0, 0 \longrightarrow K_2 \longrightarrow 0) \cong Hom(\ker T_i, K_2)$. Where (iii) will need (3) and (5) and result in $Ext(\ker T_1 \longrightarrow 0 \longrightarrow 0, 0 \longrightarrow Z_2 \longrightarrow S_2 Z_2) \cong Hom(\ker T_1, S_2)$, since S_2 is one-to-one on Z_2. Now, (iv) will require (2) and (6) and result in $Ext(Y_1 \longrightarrow T_1 Y_1 \longrightarrow 0, 0 \longrightarrow 0 \longrightarrow L_2) \cong Hom(T_1 Y_1, L_2)$. Finally, (vii) will use (4) and (6) and give $Ext(0 \longrightarrow K_1 \longrightarrow 0, 0 \longrightarrow 0 \longrightarrow L_2) \cong Hom(K_1, L_2)$. Since $U_i \xrightarrow{T_i} V_i \xrightarrow{S_i} W_i$ is made of any combination of direct sums of the above table we find $Hom(\ker T_1, K_2 \oplus Z_2) = Hom(\ker T_1, (\operatorname{Im} T_2)^C)$ and $Hom(T_1 Y_1 \oplus K_1, L_2) = Hom(\ker S_1, (\operatorname{Im} S_2)^C)$. Resulting in

$$Ext(U_1 \xrightarrow{T_1} V_1 \xrightarrow{S_1} W_1, U_2 \xrightarrow{T_2} V_2 \xrightarrow{S_2} W_2)$$

$$\cong Hom(\ker T_1, (\operatorname{Im} T_2)^C) \oplus Hom(\ker S_1, (\operatorname{Im} S_2)^C).$$

We can get a similar result for presentations of the the form $\bullet \longrightarrow \bullet$. Recall, that $Ext(V_1 \xrightarrow{T_1} W_1, V_2 \xrightarrow{T_2} W_2) \cong Hom(\ker T_1, W_2')$, where $W_2' \oplus \operatorname{Im} T_2 = W_2$. Thus we get $Ext(V_1 \xrightarrow{T_1} W_1, V_2 \xrightarrow{T_2} W_2) \cong Hom(\ker T_1, (\operatorname{Im} T_1)^C)$.

Furthermore, if we have a projective presentation, $U \xrightarrow{T} V \xrightarrow{S} W$, $Ext(U \xrightarrow{T} V \xrightarrow{S} W, -) = 0$ if and only if $Hom(\ker T, -) \oplus Hom(\ker S, -) = 0$. Which is only true if $\ker T = 0$ and $\ker S = 0$. This only occurs when T and

S are one-to-one. Dually, for $U \xrightarrow{T} V \xrightarrow{S} W$, injective:

$$\begin{aligned} & Ext(-, U \xrightarrow{T} V \xrightarrow{S} W) = 0 \\ \Leftrightarrow \quad & Hom(-, (\operatorname{Im} T)^C) \oplus Hom(-, (\operatorname{Im} S)^C) = 0 \\ \Leftrightarrow \quad & (\operatorname{Im} T)^C = 0 \text{ and } (\operatorname{Im} S)^C = 0 \end{aligned}$$

Which is only true if and only if T and S are both onto.

5.4 Ext in RepAn

Now consider the more general case of A_n.

$$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3 \longrightarrow \cdots \longrightarrow \bullet_n$$

Denoting projectives as $\mathscr{P}(i)$ for A_n, we have them as

$$\bullet^0_1 \longrightarrow \bullet^0_2 \longrightarrow \cdots \longrightarrow \bullet^F_i \xrightarrow{1} \bullet^F_{i+1} \xrightarrow{1} \cdots \xrightarrow{1} \bullet^F_{n-l} \xrightarrow{1} \bullet^F_n$$

Similarly, denote injectives as $\mathscr{J}(i)$, and they will have the form

$$\bullet^F_1 \xrightarrow{1} \bullet^F_2 \xrightarrow{1} \cdots \xrightarrow{1} \bullet^F_i \longrightarrow \bullet^0_{i+1} \longrightarrow \cdots \longrightarrow \bullet^0_{n-l} \longrightarrow \bullet^0_n$$

Now consider the quotient $\mathscr{P}(i)/\mathscr{P}(j+1)$ and denote it $\mathscr{R}(i,j)$. This would then result in

$$\bullet^0_1 \longrightarrow \bullet^0_2 \longrightarrow \cdots \longrightarrow \bullet^0_{i-1} \longrightarrow \bullet^F_i \xrightarrow{1} \bullet^F_{i+1} \xrightarrow{1} \cdots \xrightarrow{1} \bullet^F_j \longrightarrow \bullet^0_{j+1} \longrightarrow \cdots \longrightarrow \bullet^0_n$$

Giving $\mathscr{R}(1,j) = \mathscr{J}(j)$ and $\mathscr{R}(i,n) = \mathscr{P}(i)$. In particular, we get the short exact sequences

$$0 \longrightarrow \mathscr{P}(j+1) \longrightarrow \mathscr{P}(i) \longrightarrow \mathscr{R}(i,j) \longrightarrow 0$$

and

$$0 \longrightarrow \mathscr{R}(i,j) \longrightarrow \mathscr{J}(j) \longrightarrow \mathscr{J}(i) \longrightarrow 0.$$

Recall, if there exists a path from i to j, then $Hom(\mathscr{P}(i), \mathscr{P}(j)) = F$. Giving us an equivalence of categories $\mathbf{Rep}A_n \cong \mathbf{Mod}\text{-}R_n$, where R_n is the ring of upper triangular matrices defined as

$$[a_{ij}] \text{ with } a_{ij} = Hom(\mathscr{P}(i), \mathscr{P}(j))$$

where

$$Hom(\mathscr{P}(i), \mathscr{P}(j)) = \begin{cases} F & i \leq j \\ 0 & i > j \end{cases}$$

Now for some vector space, V, define $\mathscr{P}_V(i)$ by replacing the field F with V. For example $\mathscr{P}_V(3)$ with $n = 5$ would be

$$\bullet_1^0 \longrightarrow \bullet_2^0 \longrightarrow \bullet_3^V \longrightarrow \bullet_4^V \longrightarrow \bullet_5^V .$$

We can define $\mathscr{I}_V(i)$ and $\mathscr{R}_V(i,j)$ in the same way. Thus $\mathscr{P}_v(i) = \oplus_d \mathscr{P}(i)$, where d is the dimension of the vector space V.

Giving us a corresponding short exact sequence

$$0 \longrightarrow \mathscr{P}_V(j+1) \longrightarrow \mathscr{P}(i)_V \longrightarrow \mathscr{R}_V(i,j) \longrightarrow 0$$

called the projective presentation on $\mathscr{R}(i,j)$.

We can the compute Ext via projective presentations. Given a representation, W, where

$$W = \bullet^{W_1} \xrightarrow{T_1} \bullet^{W_2} \xrightarrow{T_2} \cdots \xrightarrow{T_{n-1}} \bullet^{W_n},$$

$Ext(\mathscr{R}_V(i,j), W)$ is obtained through the projective resolution of $\mathscr{R}_V(i,j)$.

$$0 \longrightarrow Hom(\mathscr{R}_V(i,j), W) \longrightarrow Hom(\mathscr{P}_V(i), W) \longrightarrow Hom(\mathscr{P}_V(j+1), W) \longrightarrow Ext(\mathscr{R}_V(i,j), W) \longrightarrow 0$$

Thus by Lemma 5.2, we know that $Hom(\mathscr{P}(i), W) \cong W_i$. Since $V = \oplus F$, we have $Hom(\mathscr{P}_V(i), W) \cong Hom(V, W_i)$. Similarly,

$Hom(\mathscr{P}_V(j+1), W) \cong Hom(V, W_{j+1})$. We then obtain the following commutative diagram.

$$\begin{array}{ccccc} Hom(\mathscr{P}_V(i), W) & \longrightarrow & Hom(\mathscr{P}_V(j+1), W) & \longrightarrow & Ext(\mathscr{R}_V(i,j), W) \\ \downarrow \wr & & \downarrow \wr & & \\ Hom(V, W_i) & \xrightarrow{T^*_{i,j}} & Hom(V, W_{j+1}) & & \end{array}$$

where $T_{i,j}$ is the composition $T_j T_{j-1} \cdots T_{i+1} T_i$ from

$$W_i \xrightarrow{T_i} W_{i+1} \xrightarrow{T_{i+1}} \cdots \xrightarrow{T_{j-1}} W_j \xrightarrow{T_j} W_{j+1}.$$

Thus $Ext(\mathscr{R}_V(i,j), W) \cong \operatorname{cok} T^*_{i,j}$.

Example 5.9 $Ext(A \longrightarrow 0 \longrightarrow 0, 0 \longrightarrow B \xrightarrow{1} B)$

The graph $A \longrightarrow 0 \longrightarrow 0$ would then be $\mathscr{R}_A(1,1)$ thus $Ext(A \longrightarrow 0 \longrightarrow 0, 0 \longrightarrow B \xrightarrow{1} B) = Ext(\mathscr{R}_A(1,1), 0 \longrightarrow B \xrightarrow{1} B)$. The graph $0 \longrightarrow B \xrightarrow{1} B$ would then make $T_{1,1} = 0$ and thus $\operatorname{cok} T^*_{1,1} = Hom(A, B)$.

$Ext(A \xrightarrow{1} A \longrightarrow 0, 0 \longrightarrow B \longrightarrow 0)$

Here we get that $Ext(A \xrightarrow{1} A \longrightarrow 0, 0 \longrightarrow B \longrightarrow 0) = Ext(\mathscr{R}_A(1,2), 0 \longrightarrow B \longrightarrow 0)$ and $T_{1,2} = I$, thus $\operatorname{cok} T^*_{1,2} = 0$. □

Now, say that $W = \mathscr{R}_W(r,s)$. Thus W is the graph

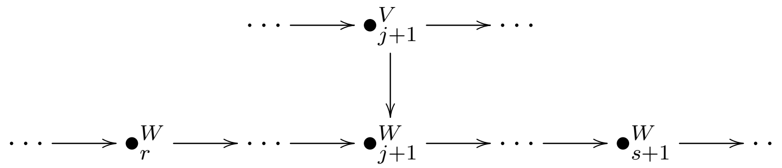
$$\bullet^0_1 \longrightarrow \bullet^0_2 \longrightarrow \cdots \longrightarrow \bullet^0_{r-1} \longrightarrow \bullet^W_r \xrightarrow{1} \bullet^W_{r+1} \xrightarrow{1} \cdots \xrightarrow{1} \bullet^W_s \longrightarrow \bullet^0_{s+1} \longrightarrow \cdots \longrightarrow \bullet^0_n$$

We then get

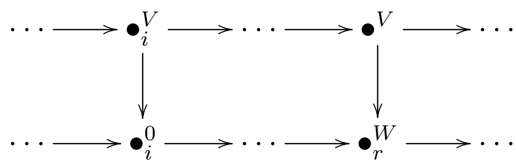
$$T_{i,j} = \begin{cases} 0 & i < r \text{ or } j < s \\ I & r \leq i \leq j \leq s \end{cases}$$

and thus making $Ext(\mathscr{R}_V(i,j), \mathscr{R}_W(r,s))$ either 0 or $Hom(V, W)$.

For Ext to be 0 we need $T_{i,j} = I$ making $Hom(V, W_i) = Hom(V, W_j)$, or $W_{j+1} = 0$ making $Hom(V, W_{j+1}) = 0$. Thus for Ext to be $Hom(V, W)$ we must have $W_{j+1} = W$ and $T_{i,j} = 0$. For $W_{j+1} = W$, we use the following diagram to illustrate the situation.



This would then force $r \leq j \leq s$. Then for $T_{i,j} = 0$ we need the following diagram.



Then forcing $i < r$. Thus

$$Ext(\mathscr{R}_V(i,j), \mathscr{R}_W(r,s)) = \begin{cases} Hom(V,W) & \text{for } \ i < r \leq j \leq s \\ 0 & \text{otherwise} \end{cases}$$

Example 5.10 A_7

We can then quickly compute any Ext group for $\mathbf{Rep}A_7$. Consider the graphs

$$0 \longrightarrow V \longrightarrow V \longrightarrow V \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

and

$$0 \longrightarrow 0 \longrightarrow W \longrightarrow W \longrightarrow W \longrightarrow 0 \longrightarrow 0.$$

These can be denoted as $\mathscr{R}_V(2,4)$ and $\mathscr{R}_W(3,5)$. Resulting in $Ext(\mathscr{R}_V(2,4), \mathscr{R}_W(3,5)) = Hom(V,W)$.

For the graphs

$$0 \longrightarrow V \longrightarrow V \longrightarrow V \longrightarrow V \longrightarrow V \longrightarrow 0$$

and

$$0 \longrightarrow 0 \longrightarrow W \longrightarrow W \longrightarrow 0 \longrightarrow 0 \longrightarrow 0,$$

we have $\mathscr{R}_V(2,6)$ and $\mathscr{R}_W(3,4)$. Giving $Ext(\mathscr{R}_V(2,6), \mathscr{R}_W(3,4)) = 0$.

Finally, consider the graphs

$$0 \longrightarrow 0 \longrightarrow V \longrightarrow V \longrightarrow V \longrightarrow 0 \longrightarrow 0$$

and

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow W \longrightarrow W \longrightarrow 0 \longrightarrow 0.$$

Here we get $\mathscr{R}_V(3,5)$ and $\mathscr{R}_W(4,5)$. Making $Ext(\mathscr{R}_V(3,5), \mathscr{R}_W(4,5)) = Hom(V,W)$.

Bibliography

[1] Dummit, David S. and Foote, Richard M. *Abstract Algebra.* Wiley 3rd Edition 2003.

[2] Hilton, P.J. and Stammbach U. *A Course in Homological Algebra.* Springer 2nd Edition 1971.

[3] Lang, Serge. *Algebra* Springer 3rd Edition 2002.

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