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GÖDEL'S INCOMPLETENESS THEOREMS

A Thesis
Presented To
Eastern Washington University
Cheney, Washington

In Partial Fulfillment of the Requirements
for the Degree
Master of Science

By
Jessica Dickson
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Abstract

Incompleteness or inconsistency? Kurt Gödel shocked the mathematical community in 1931 when he proved any effectively generated, sufficiently complex, and sound axiomatic system could not be both consistent and complete. This thesis will explore two formal languages of logic and their associated mechanically recursive proof methods with the goal of proving Gödel's Incompleteness Theorems. This, in combination with an assignment of a natural number to every string of an axiomatic system, will be used to show a consistent system contains a true statement of the form "This sentence is unprovable," and a complete system contains a proof of its own consistency only if it is inconsistent.

Contents

Chapter 1

Introduction

Prove that the axioms of arithmetic are consistent. David Hilbert posed this question in 1900 when he released his collection of 23 problems he hoped mathematicians would solve in the 20th century. Thirty years later, a young Kurt Gödel answered this query when he proved the axioms of arithmetic could not be both consistent and complete, that is, the axioms could either both prove and disprove a statement or there was a statement which was true, but unprovable.

> "One can (assuming the consistency of classical mathematics) even give examples of propositions which are reallly contextually true but unprovable in the formal system of classical mathematics." [7, pp. 155-156]

This is Gödel's First Incompleteness Theorem, a single sentence innocuously and informally announced October 7th, 1930 during a general discussion on the third and final day of a conference in Könisberg (the capital of East Prussia). At the time, Gödel was 24, a relatively unknown graduate student who gave a twenty minute lecture on the completeness of first order logic the day before. He stated his famous theorem in such a matter-of-fact way that almost everyone in attendance, with the exception of John von Neumann, underestimated its significance. The discussion itself, into which Gödel had interjected his claim, continued on as if the statement had never been made.

Although most mathematicians at the Könisberg conference reacted with incomprehension to Gödel's statement, its ramifications were eventually realized. Peano Arithmetic, Zermelo-Fraenkel set theory, group theory; almost every non-trivial axiomatic system is hampered by either inconsistency or incompleteness. By itself, Gödel's proof was remarkable. It used natural numbers to describe metamathematics (logic) and show the sentence "This very statement is not provable within the system," was true, but unprovable, all the while avoiding a paradox.

This thesis will use logic to prove Gödel's Incompleteness Theorems. The first chapter is devoted to sentential logic, a formal language consisting of true and false statements which can be combined using connectives. First order logic is an extension of sentential logic, obtained by the addition of variables, parameters, predicates and quantifiers. Models will be used to illustrate these concepts in various axiomatic systems. Semantic tableaus will prove that certain statements in logic or an axiomatic system are true. Semantic Tableaus will then be used to show both sentential and first order logic are sound and complete, that is, a statement of logic is true if and only if there exists a proof for it. In the sixth chapter, first order logic will be used to prove every axiomatic system which contains basic arithmetic also contains a sentence which is true, but unprovable, proving Gödel's First Incompleteness Theorem. His Second Incompleteness Theorem, which states the lack of a proof for consistency in such an axiomatic system, will follow from the first.

Chapter 2

Sentential Logic

This chapter is about sentential logic, a simple language whose main building block is the proposition. It is a two-valued logic, which, unlike its multi-valued counterparts, limits the truth values of its propositions to the set $\{\mathbf{T},\mathbf{F}\}$. Various propositions with different truth values can be combined via the sentential connectives to give a combined proposition with a single truth value. When they are combined according to the syntax of sentential logic, these propositions are referred to as well-formed formula (or wffs). The Unique Readability Theorem of this chapter will be show that each of these wffs has a unique truth value. Truth tables will also be used to show some wffs are always true, regardless of the individual truth values of their component propositions. Such wffs are known as tautologies.

2.1 An Introduction to Sentential Logic

Sentential Logic, also known as propositional logic, is a system of meaningful symbols which can be specifically arranged and combined with other elements of the sentential logic syllabary to create meaningful statements. This chapter will explore and explain the three groups of symbols which comprise sential logic, the first of which is the proposition.

Definition 2.1 A **proposition** p is a statement which is either true or false, but not both.

These values, true (**T**) and false (**F**), are referred to as truth values. Propositions are found in many languages, from English and Chinese to arithmetic and set theory. The symbols p, q, r, p_1, q_1, r_1, etc, are used in sentential logic to represent propositions and their associated meanings.

Example 2.2 Which of the following sentences are propositions?

p: *The capitol of Montana is Helena.*
q: $7 \div 14 = 3$
r: *Take out the garbage.*
s: *x is an even number.*
t: *This sentence is false*

The first sentence p is true and the second, q, is false, so p and q are both propositions. The third statement r is a command and cannot claim to be either true or false while the fourth, s, includes a variable x which is unknown. Thus r and s have no truth value and therefore are not propositions. However, note that if x were assigned a value, s becomes a proposition.

The truth value of the last sentence t is tricky to ascertain. ‘*This sentence is false,*” is known as the liar’s paradox. If t is true, then so is the statment “*This sentence is false,*” implying the sentence t is actually false. But if t is false, then the statement “*This sentence is false*” is untrue and t must be true. Thus t could be considered both, or neither, true and false. Regardless, it has no single truth value and thus is not a proposition. This discrepancy of truth values is indicative of a paradox. □

Definition 2.3 A **paradox** is a statement, or group of statments, which lead to a contradiction.

Paradoxes are both a vexation and a delight for mathematicians. They can illustrate a concept, indicate a flawed mathematical system, or even lead to new branches of mathematics.

Example 2.4 Let $a = b$ be a true proposition of algrebra. Multiply both sides by a to get $a^2 = ab$. Subtract b^2 from both sides for $a^2 - b^2 = ab - b^2$. Factoring then gives $(a - b)(a + b) = (a - b)b$. Divide by $(a - b)$ for the expression to have the form $a + b = b$. Now $a = b$, so the expression becomes $(b) + b = b$ or $2b = b$. Divide by b and it is evident $2 = 1$!

Reviewing the derivation, it is evident an illegal algebraic action occurred. $a = b$ implies $a - b = 0$, thus dividing by $(a - b)$ causes 'math destruction.' The paradox that arises from division by zero helps illustrate why this act is prohibited.

□

Example 2.5 In 1901, British mathematician Bertrand Russell discovered a paradox in naive set theory [14]. Russell constructed a set S that is the set of all sets that are not members of themselves (in set theory notation this reads as $S = \{s : s \notin s\}$). He then posed the question, is S a member of itself? If S is a member of itself ($S \in S$) it contradicts its' own definition, and if it is not a member of itself ($S \notin S$), then it should be a member of itself by the same definition. This became known as Russel's Paradox, and the effort to avoid it led to the creation of Russell's type theory and Zermelo-Fraenkel set thoery. □

Example 2.6 Russell illustrated his paradox using the Barber puzzle [1] which states there is a town with only one (male) barber and every man is kept shaven by either shaving himself or by going to the barber, exclusively. Equivalently, this can be stated as "*the barber shaves only those men in town who do not shave themselves*". So, who shaves the barber?

If he shaves himself, then the barber doesn't shave him. If he does not shave himself then the barber does shave him. Either way, he is still the barber and he ends up both sh aving and not shaving himself, an apparant contradiction. Note that the barber shaving himself is a self-referential statement, it is equivalent to stating that the barber shaves the barber. This notion of self-reference will arise again in the proof of Gödel's Incompleteness Theorems.

□

Although there are many, many more paradoxes that could be examined, consider instead the statement "*p: p is not provable.*" If "*not provable*" were replaced with "*false*," p would be t of example 2.2. However, while a statement which attests its' own falsity leads to a paradox, one which proclaims its' own unprovability does not. This is the essence of Kurt Gödel's first incompleteness theorem, which he introduced in 1930 at the Königsberg conference. However, before this theorem can be addressed, it is necessary to establish a firmer grounding in logic. To this end, let us turn our attention towards the second set of symbols from the sentential logic syllabary: the sentential connectives.

Definition 2.7 Given propositions p and q, the following are **sentential connectives**, or **propositional connectives**, with their associated truth values:

> The **negation** of p is the propostion "not p". Denoted $(\neg p)$, it is true only when p is false.

The **conjunction** of p and q is the proposition "p and q". Denoted $(p \wedge q)$, it is true only when both p and q are true.

The **disjunction** of p and q is the proposition "p or q". Denoted $(p \vee q)$, it is true if either p is true or q is true, or both. Here, $\vee$ is considered an "inclusive or." An "exclusive or" requires p or q be true, but not both. In example 2.6, the 'or' used was exclusive rather than inclusive. In this paper, $\vee$ will always refer to the "inclusive or."

The **implication** p implies q is the proposition "If p, then q." Denoted $(p \Rightarrow q)$, it is false only if p is true and q is false simultaneously (Thus, if p is false or q is true individually, the implication is true).

The **equivalence** of p and q is the proposition "p if and only if q". Denoted $(p \Leftrightarrow q)$ and often abbreviated to "p iff q", it is true only if both p and q have the same truth values.

Definition 2.8 A **compound proposition** is a proposition formed using sentential connectives while a **simple proposition** is one without sentential connectives.

The propositions of example 2.2 are all simple propositions, while examples 2.9 through 2.13 show how sentential connectives combine simple propositions in various ways to create compound propositions. The truth values of these propositions rely on certain unspoken assumptions, primary among them, the frame of reference in which the proposition is being interpreted. For example, if p is the propositon "$6 < 7$," there is an implicit understanding that 6 and 7 are numerical quantities and $<$ is the inequality symbol indicating the first is 'less than' the second. Similarly, the proposition *The capital of*

Montana is Helena is true because there is an implication the proposition is interpreted at the current date, rather than, for example, the year 1860 (before Helena was founded). Thus, for examples 2.9 through 2.13, truth values are evaluated based on the "obvious" assumptions of the propositions, i.e., if the snow is green, it not because green dye was spilled on it.

Example 2.9 Negation: Consider the following propositions and their negations.

p: $(1+1=3)$ $(\neg p)$: $(1+1\neq 3)$
q: $(1+1=2)$ $(\neg q)$: $(1+1\neq 2)$
r: $(1+1\neq 3)$ $(\neg r)$: $(1+1=3)$

$(\neg p)$ is true as p is false while $(\neg q)$ and $(\neg r)$ are false as q and r are true. Note that $(\neg p) = r$ and $(\neg r) = p$ so $(\neg(\neg p)) = (\neg r) = p$. This illustraties that the double negation of a proposition is the proposition itself. □

Example 2.10 Conjunction: Consider the following propositions and their conjunctions.

p_1: *Water contains hydrogen.* p_2: *Water contains oxygen.*
q_1: *Africa is a continent.* q_2: *Snow is green.*
r_1: *Stop signs are triangular.* r_2: *Stop signs are red.*
s_1: *The Earth has three moons.* s_2: *Paris is the capital of France.*

$p = (p_1 \wedge p_2)$: *Water contains hydrogen and oxygen.*
$q = (q_1 \wedge q_2)$: *Africa is a continent and snow is green.*
$r = (r_1 \wedge r_2)$: *Stop signs are triangular and red.*
$s = (s_1 \wedge s_2)$: *The Earth is square and Paris is the capital of France.*

Here only the first compound proposition p is true, while in the last three, q, r, and s, at least one of the simple propositions used with the conjuction is false.

□

Example 2.11 Disjunction: Consider the following propositions and their disjunctions.

p_1: *3 is odd.*
q_1: $\sqrt{9} = 3$
r_1: *5 is even.*
s_1: *i is real*

p_2: *7 is odd.*
q_2: $2^2 = 5$
r_2: *6 is even.*
s_2: i^2 *is positive.*

$p = (p_1 \vee p_2)$: *3 or 7 is odd.*
$q = (q_1 \vee q_2)$: $\sqrt{9} = 3$ *or* $2^2 = 5$
$r = (r_1 \vee r_2)$: 5 *or* 6 *is even.*
$s = (s_1 \vee s_2)$: *i is real or* i^2 *is positive*

Here at least one simple propostion used in the first three compound propositions p, q, and r is true so all three are true. In the last, s, both simple propositions s_1 and s_2 are false and thus the entire compound proposition is false.

□

When an implication $(p \Rightarrow q)$ occurs between two propositions p and q, p is called the **antecedent** while q is known as the **consequent.** When a person uses an implication in everyday conversation, the truth value often relies on the relationship between the antecedent and the consequent. For example, the statement "*If I don't work, then I don't have any money,*" is true based on the consequent being tacitly understood as an effect of the antecedent, not on the individual truth values of the two. In sentential logic, the statment is only true if "*If I don't work,*" is false or if "*I don't have any money,*" is true. Similarly, the statement "*If I don't work, then I am a millionaire,*" is true only if "*If I don't work,*" is false or if "*I am a millionaire,*" is true. Thus the truth value of a compound proposition does not rely on the consequent being a consequence of the antecedent.

Example 2.12 Implication: Consider the following propositions and the implication between them.

p_1: $(\sqrt{5} = 3)$
p_2: $(6 < 7)$
q_1: *6 is a prime number.*
q_2: *12 is a prime number.*
r_1: *April has 30 days*
r_2: *May has 31 days.*
s_1: *Deciduous trees lose leaves*
s_2: *Pine trees lose needles.*

$p = (p_1 \Rightarrow p_2)$: $((\sqrt{5} = 3) \Rightarrow (6 < 7))$
$q = (q_1 \Rightarrow q_2)$: *If 6 is a prime number then 12 is as well.*
$r = (r_1 \Rightarrow r_2)$: *If April has 30 days then May has 31.*
$s = (s_1 \Rightarrow s_2)$: *If deciduous trees lose their leaves then pine trees lose their needles.*

The first two compound propositions p and q are true as p_1 and q_1 are false (the truth value of p_2 and q_2 are irrelevant when the antecedents are false). In proposition r, both r_1 and r_2 are true so r is as well. In the last compound proposition s, the antecedent s_1 true but the consequent s_2 is not, so the proposition is false. □

Example 2.13 Equivalence: Consider the following propositions and the equivalence between them. Note that the phrase "*if and only if*" is abbreviated to "*iff.*"

$p = (p_2 \Leftrightarrow p_2)$: *The dodo bird is extinct iff the manatee is endangered.*
$q = (q_1 \Leftrightarrow q_2)$: *Earth is larger than the sun iff bananas are pink.*
$r = (r_1 \Leftrightarrow r_2)$: $(4 + 2 = 6) \Leftrightarrow (4 - 2 = 6)$
$s = (s_1 \Leftrightarrow s_2)$: *The U.S.'s flag is green iff Japan's flag is red and white.*

p_1: *The dodo bird is extinct.*	p_1: *The manatee is endangered.*
q_1: *Earth is larger than the sun.*	q_2: *Bananas are pink.*
r_1: $(4 + 2 = 6)$	r_2: $(4 - 2 = 6)$
s_1: *The U.S.A.'s flag is green.*	s_2: *Japan's flag is red and white.*

In the compound propositions p and q the simple propositions which comprise them have identical truth values. In the last two compound propositions r and s the simple propositions have contradicting truth values. Thus the first two compound propositions are true and the last two are false. □

The third and final set of symbols from the sential logic syllabary are the left and right parenthses ( ). These allow compound propositions to contain more than two simple propositions, indicating which propositions are associated with which connectives and the order in which their truth values should be ascertained. To find the truth value of such compound propositions, simply note that applying definition 2.7 causes two propositions and a connective to combine into a single proposition.

Example 2.14 For p, q, and r true and s false propositions, find the truth values of the following compound propositions.

1: $((p \vee q) \wedge r)$

2: $(\neg(p \Leftrightarrow q))$

3: $((p \vee q) \Rightarrow (r \wedge s))$

The parenthesis in 1 indicate the truth value of $(p \vee q)$ should be found first and then used to evaluate the remainder of the proposition. As $(p \vee q)$ is true, so is $((p \vee q) \wedge r)$. In 2, the parenthesis indicate the truth value of $(p \Leftrightarrow q)$ (true), should be negated and thus $(\neg(p \Leftrightarrow q))$ is false. The parenthesis of the last expression indicate $(p \vee q)$ is a true antecedent and $(r \wedge s)$ a false consequent, thus the implication $((p \vee q) \Rightarrow (r \wedge s))$ is false. □

Thus propositions, the propositional connectives, and the left and right parentheses form the syllabary of sentential logic. However, the truth values of simple and compound propositions remains highly subjective, so it is necessary to find more precise definitions which avoid this ambiguity.

2.2 Sentential Logic as a Mapping

The definitions to this point have been described using the English language which can be vague and misinterpreted. For example, consider the sentence *"Alaska is a cool place for a vacation."* Is this saying Alaska has a low temperature or is it indicating Alaska is a trendy location to relax? Even if the meaning is clear, the truth value of the sentence is conditional on who interprets it.

In example 2.9, we saw the double negation of a proposition is the proposition. However, in some dialects of English, this is not the case. The true propositional sentence *"Earth got a moon."* has two negations: *"Earth got no moon,"* and *"Earth ain't got a moon,"* . The negation of these negations is then *"Earth ain't got no moon,"*. But while the original proposition was true, the double negation is false in the dialect.

Implication and bijection also fall prey to misunderstandings. The sentences "*If the Earth's surface is 70.8% water then it is 29.2% land,"* and "*If the Earth's surface is 90% water then it is 29.2% land,"* are both true compound propositions, but the second sentence is false when considered as a simple proposition where the truth value depends on whether *"The Earth's surface* *is 29.2% land,"* is a valid consequence of *"The Earth's surface is 90% land."* The English language wants to create a dependence between the antecedent and the consequent that does not exist in sentential logic.

Thus it is necessary to refine the previous definitions, avoiding the ambiguity of the English language by focusing on the mapping that exists between propositions, their connectives, and the set of truth values $\{\mathbf{T}, \mathbf{F}\}$. A mapping, or a function, is an association between two sets where every element in the first set is paired with exactly one element of the second. In sentential logic, the only mapping considered is called the truth assignment, a function which sends propositions to their truth values, i.e. the map,

$$\{p, q, r, p_1, q_1, r_1, ...\} \rightarrow \{\mathbf{T}, \mathbf{F}\}$$

The truth value of a proposition p given by a truth assignment M will be denoted p_M.

Definition 2.15 A **proposition** is a symbol p and a truth assignment which yields p_M.

Thus the notion of 'understanding' whether p is true or false is circumvented; the mapping explicitely gives p a truth value p_M (although note there are multiple possible truth assignments). Propositional connectives act as operators, producing a new truth value from the truth values of one or more propositions.

Definition 2.16 A **sentential connective** is an operation which sends the truth values, given by an assignment M, of one or two propositions p and q to a single truth value as follows:

$(\neg p) : \{\mathbf{T}, \mathbf{F}\} \rightarrow \{\mathbf{F}, \mathbf{T}\}$

If $p_M = \mathbf{T}$ then $(\neg p)_M = \mathbf{F}$.

If $p_M = \mathbf{F}$ then $(\neg p)_M = \mathbf{T}$.

$(p \wedge q)$: $\{\mathbf{T}, \mathbf{F}\}$ x $\{\mathbf{T}, \mathbf{F}\} \rightarrow \{\mathbf{T}, \mathbf{F}\}$

If $p_M = \mathbf{T}$ and $q_M = \mathbf{T}$ then $(p \wedge q)_M = \mathbf{T}$.

Otherwise, $(p \wedge q)_M = \mathbf{F}$.

$(p \vee q)$: $\{\mathbf{T}, \mathbf{F}\}$ x $\{\mathbf{T}, \mathbf{F}\} \rightarrow \{\mathbf{T}, \mathbf{F}\}$

If $p_M = \mathbf{F}$ and $q_M = \mathbf{F}$ then $(p \vee q)_M = \mathbf{F}$.

Otherwise, $(p \vee q)_M = \mathbf{T}$.

$(p \Rightarrow q)$: $\{\mathbf{T}, \mathbf{F}\}$ x $\{\mathbf{T}, \mathbf{F}\} \rightarrow \{\mathbf{T}, \mathbf{F}\}$

If $p_M = \mathbf{T}$ and $q_M = \mathbf{F}$ then $(p \Rightarrow q)_M = \mathbf{F}$.

Otherwise, $(p \Rightarrow q)_M = \mathbf{T}$.

$(p \Leftrightarrow q)$: $\{\mathbf{T}, \mathbf{F}\}$ x $\{\mathbf{T}, \mathbf{F}\} \rightarrow \{\mathbf{T}, \mathbf{F}\}$

If $p_M = q_M$ then $(p \Leftrightarrow q)_M = \mathbf{T}$.

If $p_M \neq q_M$ then $(p \Leftrightarrow q)_M = \mathbf{F}$.

The first connective $\neg$ is called an unary operation as it works only on a single proposition. The other connectives are called binary operations as they apply to two propositions.

Definition 2.17 A **compound proposition** is a proposition with at least one sentential connective operation while a **simple proposition** is a proposition with no sentential connective operation.

If α is a compound proposition containing simple propositions $\{p_1, p_2, \cdots, p_k\}$ then a truth assignment is a function M_α: $\{\mathbf{T}, \mathbf{F}\}^k \rightarrow \{\mathbf{T}, \mathbf{F}\}$, where $\{\mathbf{T}, \mathbf{F}\}^k = \{\mathbf{T}, \mathbf{F}\}x\{\mathbf{T}, \mathbf{F}\}$x...x$\{\mathbf{T}, \mathbf{F}\}$ (the cross product of k sets of $\{\mathbf{T}, \mathbf{F}\}$).

Example 2.18 For p, q, and r propositions, the following are truth assignments for the associated compound proposition.

$(\neg p)$: $\{\mathbf{T}, \mathbf{F}\} \rightarrow \{\mathbf{F}, \mathbf{T}\}$

$(p \vee q)$: $\{\mathbf{T}, \mathbf{F}\}$ x $\{\mathbf{T}, \mathbf{F}\} \rightarrow \{\mathbf{T}, \mathbf{F}\}$

$((p \vee q) \Rightarrow r)$: $\{\mathbf{T}, \mathbf{F}\}$ x $\{\mathbf{T}, \mathbf{F}\}$ x $\{\mathbf{T}, \mathbf{F}\} \rightarrow \{\mathbf{T}, \mathbf{F}\}$

$(p \vee p \vee \cdots \vee p)$: $\{\mathbf{T}, \mathbf{F}\}$ x $\{\mathbf{T}, \mathbf{F}\}$ x $\cdots$ x $\{\mathbf{T}, \mathbf{F}\} \rightarrow \{\mathbf{T}, \mathbf{F}\}$

Although the ambiguity surrounding sentential logic has begun to dissipate with these definitions, there is still an elephant in the room. As sentential logic is merely a collection of symbols, why do these propositions have meaning? In what order can the symbols of sential logic be arranged and still have a truth assignment? To answer these questions it is necessary to explore the syntax of sentential logic.

2.3 Syntax

With an infinite number of disctinct symbols available in sentential logic, it is necessary to develop a formal grammar, or syntax, which allows a mathematician to distinguish valid propositions from gibberish. Just as ”*red roses are violets blue and*” is incomprehensible in English, $p)(q\vee$ is nonsensical in sentential logic. Syntax specifies the order in which various symbols can be arranged and allows the meaning of various phrases and expressions to be consistently interpreted. Such arrangements of symbols, whether adhering to syntax or not, are called strings.

Definition 2.19 A **string** is a finite sequence of symbols taken from a specific set of symbols (a syllabary).

Example 2.20 The following are strings in their associated alphabets.

elephant, *abc*, and *brugtsx* in the English alphabet.
1+3=4, 6-2=100, and $= 896-$ in arithmetic.
$\alpha \in \Gamma$, $\emptyset$, and $A \cap B \supset$ in set theory.
$p)(q\vee$, $(p \vee q)$, and $p \vee q$ in sentential logic.

□

Definition 2.21 The **length** of a string $S{=}s_1s_2...s_n$ (where s_i is a symbol or letter) is the number n. Note that a length of 0 represents the empty string.

The length of strings may seem inconsequential, but is vital in many proofs of this thesis that use induction.

Example 2.22 Find the lengths (L) of the strings in example 2.20.

Alphabet	String	L	String	L	String	L
English	*elephant*	8	*abc*	3	*brugtsx*	7
Arithmetic	$1+3=4$	5	$6-2=100$	5	$=896-$	6
Set Theory	$\alpha\in\Gamma$	3	$\emptyset$	1	$A\cap B\supset$	4
Sentential Logic	$p)(q\vee$	5	$(p\vee q)$	5	$p\vee q$	3

□

Definition 2.23 A **formal language** is a set of finite strings defined using rules of formation (also called the formal grammer or syntax of the language).

Languages such as English or French have no set rules of formation and thus are not formal languages. Naive set theory, or Cantor's set thoery, as it relies on natural language to describe sets, is also not a formal language. Formal languages do not have to be deeply developed and may contain only a finite number of strings.

Example 2.24 Consider the following constructed formal language with the alphabet { c, s, r, m, is, on, ad, at, -} and formal grammer:

1: A string must have a length of seven.
2: The third and fifth symbol must be ” -”.
3: A symbol from the set {ad, at} must be directly preceded by a symbol from the set {c, s, r, m}

Condsider the following strings:

○ c a t - i s - s a d
○ s a t - o n - r a t
○ r a t - i s - f l a t
○ m a d - a t - c a t
○ c - - - - - - - m

The first two strings follow the rules of formation in this language. The third has a symbol not in the alphabet, "fl". The fourth violates rule 3 as "at" is preceded by "-". The last follows the rules of formation. Note that there is no inherent meaning to any of the strings until it is assigned by the user, i.e. "c a t - i s - s a d" and "c - - - - - m" are ‘equally’ significant.

Sentential logic can now be defined as a formal language with an infinite number of distinct symbols, including propositions, propositional connectives, and the left and right parenthesis from which, following the rules of formation, strings called well-formed formula can be constructed .

Definition 2.25 A **well-formed formula**, or **wff**, is a string which follows the syntax of sentential logic, i.e. can be obtained by applying a finite number of times the following rules of formation:

p: If p is a proposition, then p is a wff.
$\neg$: If α is a wff, then $(\neg\alpha)$ is a wff.
$\wedge$: If α and β are wffs, then $(\alpha \wedge \beta)$ is a wff.
$\vee$: If α and β are wffs, then $(\alpha \vee \beta)$ is a wff.
$\Rightarrow$: If α and β are wffs, then $(\alpha \Rightarrow \beta)$ is a wff.
$\Leftrightarrow$: If α and β are wffs, then $(\alpha \Leftrightarrow \beta)$ is a wff.

(Note that this is a recursive definition so additional wffs can be constructed from the newly formed ones above.)

Definition 2.26 A **simple wff** is a proposition while a **compound wff** is a wff formed by applying the rules of formation to at least one other wff

Example 2.27 For α and β wffs, the following are compound wffs.

- $(\neg\alpha)$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \Rightarrow \beta)$, and $(\alpha \Leftrightarrow \beta)$.
- $((\neg\alpha) \Rightarrow \beta)$
- $((\alpha \vee \beta) \Leftrightarrow (\beta \Rightarrow \alpha))$

□

Definition 2.28 A **parsing sequence** of a wff α is a series of strings created by repeatedly applying the rules of formation in a recursive manner. A string which is not a wff does not have a parsing sequence.

Example 2.29 Consider the last line of strings from example 2.20 with p and q propositions. It is evident that $p)(q\vee$ cannot be created using the rules of formation while $(p \vee q)$ can and has the parsing sequence

1: p and q are wffs by the first rule of formation.
2: $(p \vee q)$ is a wff by $\vee$ and 1.

However $p\vee q$, although very similar to $(p\vee q)$, does not have a parsing sequence and is therefore not a wff.

□

These missing parenthesis may seem inconsequential (after all, it is easily understood $p \vee q$ is equivalent to $(p \vee q)$), but their location can vastly change the meaning of a sentence. For example $((p \vee q) \wedge r) \Rightarrow s \vee t$ could be either $((p \vee q) \wedge r) \Rightarrow (s \vee t)$ or $(((p \vee q) \wedge r) \Rightarrow s) \vee t$. For this reason there is an understood "order of operations" that indicates which wffs should be given precedence and allows the dropping of excessive parenthesis.

Definition 2.30 The **order of operations** for a wff is listed (in descending importance) below:

(,): Strings inside parenthesis
$\neg$: Negation
$\vee, \wedge$: Disjunction and Conjunction
$\Rightarrow, \Leftrightarrow$: Implication and Equivalence

Thus it is evident $((p \vee q) \wedge r) \Rightarrow s \vee t$ is equivalent to $((p \vee q) \wedge r) \Rightarrow (s \vee t)$ rather than $(((p \vee q) \wedge r) \Rightarrow s) \vee t$. Wffs which are simplified using the order of operations are often called abbreviated wffs.

Example 2.31 Consider the abbreviated wff $\neg p \wedge q \Leftrightarrow r \wedge (s \vee t)$. Using the order of operations, work backwards to find its original form

Parenthesis: $\neg p \wedge q \Leftrightarrow r \wedge (s \vee t)$
Negations: $(\neg p) \wedge q \Leftrightarrow r \wedge (s \vee t)$
Disjunction and Conjunction: $((\neg p) \wedge q) \Leftrightarrow (r \wedge (s \vee t))$
Implication: $(((\neg p) \wedge q) \Leftrightarrow (r \wedge (s \vee t)))$

However, it is interesting to note if the string had been $\neg p \wedge q \Leftrightarrow r \wedge s \vee t$ it would have been impossible to recapture the original form as it is unclear whether $r \wedge s \vee t$ is $(r \wedge s) \vee t$ or $r \wedge (s \vee t)$. Thus it is important to use parenthesis to avoid confusion when two connectives are on the same "level" of order of operations and can potentially apply to the same wffs. $\square$

Example 2.32 Find the parsing sequence for the abbreviated wff $\neg p \wedge q \Leftrightarrow r \vee (s \vee t)$

1: p, q, r, s, and t are wffs by the first rule of formation.
2: $(s \vee t)$ is a wff by $\vee$ and 1.
3:$(\neg p)$, and thus $\neg p$, is a wff by $\neg$ and 1.

4: $((\neg p) \wedge q)$, and thus$\neg p \wedge q$ is a wff by $\wedge$, 1, and 3.
5: $(r \vee (s \vee t))$, and thus $r \wedge (s \vee t)$ is a wff by $\vee$, 1, and 2.
6: $(((\neg p) \wedge q) \Leftrightarrow (r \vee (s \vee t)))$, and thus $\neg p \wedge q \Leftrightarrow r \vee (s \vee t)$ is a wff by $\Leftrightarrow$, 4, and 5.

However, this is not the only parsing sequence for this wff; the following sequence is equally valid

1: p, q, r, s, and t are wffs by the first rule of formation.
2: $(\neg p)$, and thus $\neg p$, is a wff by $\neg$ and 1.
3:$(s \vee t)$ is a wff by $\vee$ and 1.
4: $(r \vee (s \vee t))$, and thus $r \vee (s \vee t)$ is a wff by $\vee$, 1, and 3.
5: $((\neg p) \wedge q)$, and thus$\neg p \wedge q$ is a wff by $\wedge$, 1, and 2.
6: $(((\neg p) \wedge q) \Leftrightarrow (r \vee (s \vee t)))$, and thus $\neg p \wedge q \Leftrightarrow r \vee (s \vee t)$ is a wff by $\Leftrightarrow$, 4, and 5.

□

Differences in parsing sequences arise from the order in which we evaluate component wffs. For example, given a string $(\alpha_1 *_\alpha \alpha_2) * (\beta_1 *_\beta \beta_2)$, where α_1, α_2, β_1, and β_2 are wffs and $*_\alpha, *_\beta$, and $*$ are binary connectives, there are two possible parsing sequences:

1: α_1, α_2, β_1, and β_2 are wffs by hypothesis.
2: $(\alpha_1 *_\alpha \alpha_2)$ is a wff by 1 and $*_\alpha$
3: $(\beta_1 *_\beta \beta_2)$ is a wff by 1 and $*_\beta$
4: $(\alpha_1 *_\alpha \alpha_2) * (\beta_1 *_\beta \beta_2)$ is a wff by 3, 4, and $*$.

and

1: α_1, α_2, β_1, and β_2 are wffs by hypothesis.
3: $(\beta_1 *_\beta \beta_2)$ is a wff by 1 and $*_\beta$
2: $(\alpha_1 *_\alpha \alpha_2)$ is a wff by 1 and $*_\alpha$
4: $(\alpha_1 *_\alpha \alpha_2) * (\beta_1 *_\beta \beta_2)$ is a wff by 3, 4, and $*$.

Each parsing sequence originated from a different order in which the wffs joined by a binary connective were evaluated. Thus many different parsing sequences often exist for the same wff. This raises the worry that there may be numerous interpretations of a single wff as well. Now that parsing sequences gives us a step by step look at a wff, it is tempting to jump right in and assign truth value. However, it is evident different parsing sequences exist for the same wff. Thus it must first be proven that every wff of sentential logic can only be interpreted in one way; that it is not possible to get two different truth values for a wff from a single truth assignment.

2.4 Uniqueness of Truth Values

Many of the proofs in this section rely heavily on induction of the length of wffs, including the Unique Readability Theorem, which proves the uniqueness of a wff's truth value. Once such uniqueness has been proven, truth values can be assigned to a wff based on the truth values of its individual components.

Definition 2.33 A wff is **balanced** if it has the same number of left brackets as right brackets.

Theorem 2.34 *Every wff is balanced.*

Proof: If a wff has length 0 it is the empty string and thus (trivially) the wff is balanced. If a wff has lenth 1, then it must be a proposition and has no brackets so is balanced. Assume every wff of length n or less is balanced and suppose α is a wff of lenth $n+1$. Then α is in one of two forms: If $\alpha = (\neg\beta)$ then β has a length of at most n and is balanced by assumption. α then has one more of each bracket than β and hence is also balanced. If

$\alpha = (\beta_1 * \beta_2)$ where $*$ is binary connective then β_1 and β_2 are wffs of length n or less and balanced. Then the number of left brackets in β_1 and β_2 plus 1 is equal to the number of total left brackets in α. The number of right brackets is the same, so α is balanced. Thus, by induction, all wffs are balanced. ∎

Definition 2.35 A string s is an **initial part** of another string p if s is formed by removing one or more symbols at the end of p.

Example 2.36 Consider the wff $((p \vee q) \wedge r)$. The initial parts of α are the empty string and the strings

$($, $(($, $((p$, $((p\vee$, $((p \vee q$, $((p \vee q)$, $((p \vee q)\wedge$, $((p \vee q) \wedge r$

However, note that none of the above strings are wffs, which leads to theorem 2.37.

Theorem 2.37 *Any string which is an initial part of a wff is not a wff.*

Proof: Consider a wff α of length n. If $n = 0$ then α is the empty string and not a wff. If $n = 1$ the initial part of α must be the empty string and thus is not a wff. Assume that any initial part of a wff with length at most n is not a wff. Let α be a wff of length $n + 1$. Then α is either $(\neg\beta)$ or $(\beta_1 * \beta_2)$ where $*$ is a binary connective.

Case 1: α is $(\neg\beta)$. Let it be assumed an initial part s of α is a wff, and attempt to find a contradiction. If s is a wff then α is s followed by some non-empty string t (written $\alpha = st$ for convenience). Now s is either the single left parenthesis (and thus not a wff and thus a contradiction) or begins with $(\neg$. So $s = (\neg u)$ where u is a wff. So $(\neg\beta) = \alpha = st = (\neg u)t$ becomes, by removing the initial $\neg$ and parenthesis, $\beta = ut$. But then β is a wff with length at most n and has the wff u as an initial part, contradicting the induction hypothesis that any initial part of a wff with length at most n is not a wff. Thus s cannot be a wff.

Case 2: α is $(\beta_1 * \beta_2)$. Assume an initial part s of α is a wff (again looking for contradiction). Then $\alpha = st$ where t is not the empty string. Now s begins with "(", so either $s = (\neg s_1)$ or $s = (s_1 \circ s_2)$ for $\circ$ a binary connective, and wffs s_1 and s_2.

Subcase 2.1: s is $(\neg s_1)$. Then $(\beta_1 * \beta_2) = (\neg s_1)t$, and by removal of the first parenthesis, $\beta_1 * \beta_2) = \neg s_1)t$. But then β_1 begins with a $\neg$ and can't be a wff (the rules of formation don't allow it). Thus s cannot be a wff.

Subcase 2.2: s is $(s_1 \circ s_2)$. Then $(\beta_1 * \beta_2)$ is $(s_1 \circ s_2)t$ and by removing the initial "(", this becomes $\beta_1 * \beta_2) = s_1 \circ s_2)t$. Both β_1 and s_1 are wffs with length at most n, so by the induction hypothesis, neither can be an initial part of the other. However, they begin at the same place within α, so they can only be the same. Thus $\beta_1 * \beta_2) = \beta_1 \circ s_2)t$ which implies $* = \circ$ and $\beta_2) = s_2)t$. But then the wff s_2 is an initial part of the wff β_2 of length at most n, giving a contradition to the induction hypothesis. Hence s is not a wff.

Thus, by induction, any string which is an inital part of a wff is not a wff. ∎

Definition 2.38 The **main connective** of a wff α is the connective used in the last step of its parsing sequence.

If α, β_1 and β_2 are wffs and $*$ is a binary connective, then if $\alpha = (\beta_1 * \beta_2)$, $*$ is the main connective.

Example 2.39 The main connective for the wff $\neg p \wedge q \Leftrightarrow r \vee (s \vee t)$ is $\Leftrightarrow$ as by example 2.32, it is the last connective used in step 6 of its parsing sequence.

□

If α is a simple wff then it does not have a sentential connective and thus does not have a main connective either. Note that a simple wff is a simple proposition, and does not start with a left parenthesis.

Theorem 2.40 The Unique Readability Theorem: *Every wff α which starts with a left parenthesis has exactly one main connective.*

Proof: Then α is of the form $(\neg\beta)$ or $(\beta_1 * \beta_2)$ where $*$ is a binary connective and β, β_1, and β_2 are wffs.

Case 1: $\alpha = (\neg\beta)$. Then its parsing sequence has only two steps; 1: β is a wff by hypothesis and 2: $(\neg\beta)$ is a wff by $\neg$. Thus $\neg$ must be the main connective.

Case 2: $\alpha = (\beta_1 * \beta_2)$. Suppose also that $\alpha = (\delta_1 \circ \delta_2)$ where δ_1 and δ_2 are wffs and $\circ$ is a binary connective. The goal is to suppose there are two main connectives $*$ and $\circ$ and show they are the same. Then β_1 and δ_1 must start at the same place (after the first left bracket). But by theorem 2.37, neither β_1 or δ_1 can be an intial part of the other or they wouldn't be wffs. Thus $\beta_1 = \delta_1$, $* = \circ$, and $\beta_2 = \delta_2$. ∎

This theorem implies every wff is uniquely interpreted, as every wff which is not a simple proposition can be broken down into shorter wffs in only one way. For example, consider a wff α which begins with a left parenthesis (i.e., is not a simple proposition). The Unique Readability Theorem says there is exactly one main connective, so $\alpha = (\alpha_1 * \alpha_2)$ or $\alpha = (\neg\beta)$. Either way, every new wff which is not a simple proposition breaks down into smaller wffs combined via a main connective. There is no other options to break down the wffs, so α can only be read in one way. Thus for every model, or truth assignment, there is only one truth value per wff.

Definition 2.41 A **model** M is a function which assigns every proposition p a truth value p_M, i.e. a mapping $M : \{p_1, p_2, ...\} \rightarrow \{\mathbf{T}, \mathbf{F}\}$.

In other words, the truth values of propositions depend on the model in which they exist. In an alternative dimension M, where swine have wings, the proposition *p: pigs fly* takes a different truth value. Often this is denoted $p_M = \mathbf{T}$ to indicate a certain proposition is true in a given model, or untrue in a model by $p_M \neq \mathbf{T}$ or $p_M = \mathbf{F}$.

Example 2.42 Consider the commutative property $a{*}b = b{*}a$. While always true in certain models such as :

$*$: Addition of real numbers: $x + y = y + x$
$*$: Multiplication of complex numbers: $(a+bi)\cdot(c+di) = (c+di)\cdot(a+bi)$
$*$: Dot product of vectors: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
$*$: Z_6 the integers mod 6

It can be false in other models:

$*$: Subtraction of real numbers: $x - y \neq y - x$
$*$: Division of complex numbers: $(a + bi) \div (c + di) \neq (c + di) \div (a + bi)$
$*$: Cross product of vectors: $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$
$*$: S_3 the symmetric group of order 3

Definition 2.43 To find the truth value α_M of a wff α in a model M, simply use the following rules:

p: If α is a proposition p, then $\alpha_M = p_M$.

$\neg$: If $\alpha_M = \mathbf{T}$ then $(\neg\alpha)_M = \mathbf{F}$.
If $\alpha_M = \mathbf{F}$ then $(\neg\alpha)_M = \mathbf{T}$.

$\wedge$: If $\alpha_M = \mathbf{T}$ and $\beta_M = \mathbf{T}$ then $(\alpha \wedge \beta)_M = \mathbf{T}$.
Otherwise, $(\alpha \wedge \beta)_M = \mathbf{F}$.

$\vee$: If $\alpha_M = \mathbf{F}$ and $\beta_M = \mathbf{F}$ then $(\alpha \vee \beta)_M = \mathbf{F}$.
Otherwise, $(\alpha \vee \beta)_M = \mathbf{T}$.

$\Rightarrow$: If $\alpha_M = \mathbf{T}$ and $\beta_M = \mathbf{F}$ then $(\alpha \Rightarrow \beta)_M = \mathbf{F}$.
Otherwise, $(\alpha \Rightarrow \beta)_M = \mathbf{T}$.

$\Leftrightarrow$: If $\alpha_M = \beta_M$ then $(\alpha \Leftrightarrow \beta)_M = \mathbf{T}$.
If $\alpha_M \neq \beta_M$ then $(\alpha \Leftrightarrow \beta)_M = \mathbf{F}$.

Thus given a wff, one can simply find a parsing sequence and apply one of the above rules at each step.

Example 2.44 Find the truth value of $\neg p \wedge q \Leftrightarrow r \vee (s \vee t)$ for a model M where $p_M = \mathbf{T}$, $q_M = \mathbf{F}$, $r_M = \mathbf{F}$, $s_M = \mathbf{T}$, and $t_M = \mathbf{F}$. To do this, we use the parsing sequence found in example 2.32, and evaluate the truth value at each step.

1: $p_M = \mathbf{T}$, $q_M = \mathbf{F}$, $r_M = \mathbf{F}$, $s_M = \mathbf{T}$, and $t_M = \mathbf{F}$ by hypothesis.
2: $(s \vee t)_M = \mathbf{T}$ by $\vee$ and 1.
3: $(\neg p)_M = \mathbf{F}$ by $\neg$ and 1.
4: $(\neg p \wedge q)_M = \mathbf{F}$ by $\wedge$, 1, and 3.
5: $(r \vee (s \vee t))_M = \mathbf{T}$ by $\vee$, 1, and 2.
6: $(\neg p \wedge q \Leftrightarrow r \vee (s \vee t))_M = \mathbf{F}$ by $\Leftrightarrow$, 4, and 5.

□

Consider the same wff in another model where p_M and q_M are switched.

Example 2.45 Find the truth value of $\neg p \wedge q \Leftrightarrow r \vee (s \vee t)$ for a model M where $p_M = \mathbf{F}$, $q_M = \mathbf{T}$, $r_M = \mathbf{F}$, $s_M = \mathbf{T}$, and $t_M = \mathbf{F}$. With the same parsing sequence, evaluating the truth values gives:

1: $p_M = \mathbf{F}$, $q_M = \mathbf{T}$, $r_M = \mathbf{F}$, $s_M = \mathbf{T}$, and $t_M = \mathbf{F}$ by hypothesis.
2: $(s \vee t)_M = \mathbf{T}$ by $\vee$ and 1.

3: $(\neg p)_M = \mathbf{T}$ by $\neg$ and 1.
4: $(\neg p \wedge q)_M = \mathbf{T}$ by $\wedge$, 1, and 3.
5: $(r \vee (s \vee t))_M = \mathbf{T}$ by $\vee$, 1, and 2.
6: $(\neg p \wedge q \Leftrightarrow r \vee (s \vee t))_M = \mathbf{T}$ by $\Leftrightarrow$, 4, and 5.

□

Theorem 2.46 *Let α be a wff in a model M. Then the truth value α_M is the same for all parsing sequences of α.*

Proof: This theorem establishes that every wff has a unique truth value, and can be proven by induction on the length of wffs. Let α be a wff of length n. If $n = 0$, then α is a simple wff, i.e. a simple proposition, and has a 1 step parsing sequence. Thus, there can be only a single truth value for this parsing sequence. Assume any wff of length n or less has the same truth value for every parsing sequence (i.e., its truth value is unique) and consider the wff α of length $n+1$. Then either $\alpha = (\neg\beta)$ or $\alpha = (\beta_1 * \beta_2)$, where $*$ is a binary connective.

Case 1: $\alpha = (\neg\beta)$. Then β is of length at most n and, thus by induction hypothesis, has the same truth value for each of its parsing sequences. The parsing sequence for α is found by adding the same single step to any parsing sequence for β, and therefore must have the same truth value in every parsing sequence for α.

Case 2: $\alpha = (\beta_1 * \beta_2)$. By theorem 2.40, $*$ is the main connective of α and thus occurs in the last step of the parsing sequence for α. Now β_1 and β_2 have lengths at most n and therefore, by the induction hypothesis, have the same truth value for each of their parsing sequences. Thus, as $*$ occurs in the last step of the parsing sequence for α, and β_1 and β_2 have unique truth values, by definition 2.43, α must also have a unique truth value.

Thus, by induction, the truth value of a wff is the same for all of its parsing sequences. ▮

Definition 2.47 A model M **models**, or **satisfies**, a wff α if $\alpha_M = \mathbf{T}$. M **models** a set of wffs H, if M satisfies every wff in H. These are denoted $M \models \alpha$ and $M \models H$, respectively.

Conversely, one can say M does not model a wff α if $\alpha_M = \mathbf{F}$ and denote this by $M \not\models \alpha$. Rules for a wff satisfying a model M are given below.

Proposition 2.48 If M is a model and α and β are wffs, then:

$\neg\neg$: If $M \models \neg\neg\alpha$, then $M \models \alpha$.
$\wedge$: If $M \models (\alpha \wedge \beta)$, then $M \models \alpha$ and $M \models \beta$,
$\neg\wedge$: If $M \models \neg(\alpha \wedge \beta)$, then $M \models \neg\alpha$ or $M \models \neg\beta$.
$\vee$: If $M \models (\alpha \vee \beta)$, then either $M \models \alpha$ or $M \models \beta$.
$\neg\vee$: If $M \models \neg(\alpha \vee \beta)$, then $M \models \neg\alpha$ and $M \models \neg\beta$.
$\Rightarrow$: If $M \models (\alpha \Rightarrow \beta)$, then either $M \models \neg\alpha$ or $M \models \beta$.
$\neg \Rightarrow$: If $M \models \neg(\alpha \Rightarrow \beta)$, then either $M \models \alpha$ or $M \models \neg\beta$.
$\Leftrightarrow$: If $M \models (\alpha \Leftrightarrow \beta)$, then either $M \models (\alpha \wedge \beta)$ or $M \models \neg(\alpha \wedge \beta)$.
$\neg \Leftrightarrow$: If $M \models \neg(\alpha \Leftrightarrow \beta)$, then $M \models \neg(\alpha \Rightarrow \beta)$ or $M \models \neg(\beta \Rightarrow \alpha)$.

Proof: The reasoning for this proposition follows from definition 2.43 and 2.47. For example, $M \models \neg\neg\alpha$ means $(\neg\neg\alpha)_M = \mathbf{T}$, so by 2.43, $(\neg\alpha)_M = \mathbf{F}$ and thus $\alpha_M = \mathbf{T}$ so $M \models \alpha$. Likewise, $M \models (\alpha \wedge \beta)$ means $(\alpha \wedge \beta)_M = \mathbf{T}$ and thus $\alpha_M = \beta_M = \mathbf{T}$ so $M \models \alpha$ and $M \models \beta$. $\neg\wedge$ and $\neg\vee$ follow from similar reasoning and DeMorgan's Laws. The reasoning for the rest is much the same and is left to the reader. ▮

However, as the number of component propositions in a wff increases, the complexity of finding its truth value in a model does as well. As this evaluation is straightforward, the process and proof of a truth value can be arranged into table form. The alternate problem of finding a model which **satisfies** a wff, that is, makes it true, can also be solved via table form.

2.5 Truth Tables

Truth tables are a easy way to organize and easily review the truth values of wffs in various models. As the various propositions under consideration increase, the number of possible models grows even quicker. For a single proposition p, there are only two possible models, one in which p is true and one in which it is false. For two propositions p and q, the number of models double:

$p_{M_1} = \mathbf{T}$, $q_{M_1} = \mathbf{T}$
$p_{M_2} = \mathbf{T}$, $q_{M_2} = \mathbf{F}$
$p_{M_3} = \mathbf{F}$, $q_{M_3} = \mathbf{T}$
$p_{M_4} = \mathbf{F}$, $q_{M_4} = \mathbf{F}$

Similarly, a third proposition r again doubles the number of models:

$p_{M_1} = \mathbf{T}$, $q_{M_1} = \mathbf{T}$, $r_{M_1} = \mathbf{T}$
$p_{M_2} = \mathbf{T}$, $q_{M_2} = \mathbf{F}$, $r_{M_2} = \mathbf{T}$
$p_{M_3} = \mathbf{F}$, $q_{M_3} = \mathbf{T}$, $r_{M_3} = \mathbf{T}$
$p_{M_4} = \mathbf{F}$, $q_{M_4} = \mathbf{F}$, $r_{M_4} = \mathbf{T}$
$p_{M_5} = \mathbf{T}$, $q_{M_5} = \mathbf{T}$, $r_{M_5} = \mathbf{F}$
$p_{M_6} = \mathbf{T}$, $q_{M_6} = \mathbf{F}$, $r_{M_6} = \mathbf{F}$
$p_{M_7} = \mathbf{F}$, $q_{M_7} = \mathbf{T}$, $r_{M_7} = \mathbf{F}$
$p_{M_8} = \mathbf{F}$, $q_{M_8} = \mathbf{F}$, $r_{M_8} = \mathbf{F}$

As the number of models grow it becomes difficult to track various truth values. Truth tables alleviate some of this difficulty, providing a visual summary that is easily understood and ensures that every possible model has been considered. The above models for the two and three part propositions are dipicted as tables below:

	p	q
M_1	T	T
M_2	T	F
M_3	F	T
M_4	F	F

	p	q	r
M_1	T	T	T
M_2	T	F	T
M_3	F	T	T
M_4	F	F	T
M_5	T	T	F
M_6	T	F	F
M_7	F	T	F
M_8	F	F	F

The sentential connectives can also be easily organized into truth table form, as evidenced below.

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Example 2.49 Consider $((p \vee q) \wedge (r \Rightarrow s))$. Evaluating this wff for all 16 models would be onerous. It would be easy to skip a model without realizing it or confuse truth values when switching from model to model. With its truth table below, however, every aspect of every model is laid out and easy

to reference.

p	q	r	s	$p \vee q$	$r \Rightarrow s$	$((p \vee q) \wedge (r \Rightarrow s))$
T	T	T	T	T	T	T
T	T	T	F	T	F	F
T	T	F	T	T	T	T
T	T	F	F	T	T	T
T	F	T	T	T	T	T
T	F	T	F	T	F	F
T	F	F	T	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	T	F	T	F	F
F	T	F	T	T	T	T
F	T	F	F	T	T	T
F	F	T	T	F	T	F
F	F	T	F	F	F	F
F	F	F	T	F	T	F
F	F	F	F	F	T	F

The models which satisify $((p \vee q) \wedge (r \Rightarrow s))$ are those in which its truth value is true. For example, the model where $p_M = q_M = r_M = s_M = T$ satisfies the wff. Similarly, the model where $p_M = q_M = s_M = T$ and $r_M = F$ satisifies the wff.

□

Example 2.50 Consider DeMorgan's laws which state that *the negation of a conjunction is the disjunction of the negations* $(\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q))$ and that *the negation of a disjunction is the conjunction of the negations* $(\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q))$. Both laws can be organized into truth tables as shown below.

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$(\neg p \wedge \neg q)$	$\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$
T	T	F	F	T	F	F	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	T	F	T	T	T

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$(\neg p \vee \neg q)$	$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

The last column of both tables indicate the laws hold in every model (every model satisfies the laws). When this happens, it is reffered to as a tautology.

□

Definition 2.51 A wff α is called a **tautology** if it is true in every model, i.e. $M \models \alpha$ for every model M.

To check whether a wff α is a tautology, create a truth table which evaluates the truth value of α for every model.

Example 2.52 Are either of the wffs $((p \vee q) \Rightarrow (q \wedge p))$ or $((q \wedge p) \Rightarrow (p \vee q))$ tautologies?

p	q	$(p \vee q)$	$(q \wedge p)$	$((p \vee q) \Rightarrow (q \wedge p))$	$((q \wedge p) \Rightarrow (p \vee q))$
T	T	T	T	T	T
T	F	T	F	F	T
F	T	T	F	F	T
F	F	F	F	T	T

It is evident from the truth table that the wff $((p \vee q) \Rightarrow (q \wedge p))$ is false in two models and so is not a tautology. However the second wff $((q \wedge p) \Rightarrow (p \vee q))$ is true in every model and thus is a tautology. $\square$

A truth table which gives a tautology can also show whether the reasoning which comprises a proof, also called an argument form, is justifiable. For example, modus ponens and modus tollens are argument forms which are tautologies. Modus ponens, which is latin for "the way that affirms by affirming," states that if the statement $p \Rightarrow q$ is true, and if p is true, then q must be true as well. For example, consider the compound proposition "*If I don't work, then I don't have any money,*." If the statement and its antecedent are both true, then so is the consequent "*I don't have any money.*" Modus ponens is also implicit in many mathematical proofs, including those in this chapter. For example, in theorem 2.46, it was reasoned $*$ was the main connective of $\alpha = (\beta_1 * \beta_2)$ by the following modus ponens argument:

p: $\alpha = (\beta_1 * \beta_2)$

q: $*$ *is the main connective of* $\alpha = (\beta_1 * \beta_2)$

$p \Rightarrow q$: $\alpha = (\beta_1 * \beta_2)$ *implies* $*$ *is the main connective of* $\alpha = (\beta_1 * \beta_2)$

p: True by case assumption.

$p \Rightarrow q$: True by theorem 2.40.

q: True by modus ponens.

On the other hand, modus tollens, which is latin for "the way that denies by denying," states that if a statement $p \Rightarrow q$ is true, and q is false, then p is false as well. For example, Let p be the proposition "*a+b is odd*" and q the proposition "*a is odd or b is odd, but not both.*" Then, if $p \Rightarrow q$: "*If a+b is odd then a is odd or b is odd, but not both*" is true, but q is false, then p is false as well. Thus if a and b are both odd or both we can conclude $a + b$ is even. Both modus ponens and modus tollens can be proven via a truth table.

Example 2.53 Proof of modus ponens and modus tollens.

Modus Ponens $((p \Rightarrow q) \wedge p) \Rightarrow q$

p	q	$p \Rightarrow q$	$((p \Rightarrow q) \wedge p)$	$((p \Rightarrow q) \wedge p) \Rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Modus Tollens $((p \Rightarrow q) \wedge \neg q) \Rightarrow \neg p$

p	q	$\neg p$	$\neg q$	$(p \Rightarrow q)$	$((p \Rightarrow q) \wedge \neg q)$	$(((p \Rightarrow q) \wedge \neg q) \Rightarrow \neg p)$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

□

Example 2.54 Proof that an implication and its contrapositive are equivalent, i.e., $(\alpha \Rightarrow \beta) \leftrightarrow (\neg\beta \Rightarrow \neg\alpha)$ is a tautology.

α	β	$\neg\alpha$	$\neg\beta$	$(\alpha \Rightarrow \beta)$	$(\neg\beta \Rightarrow \neg\alpha)$	$(\alpha \Rightarrow \beta) \leftrightarrow (\neg\beta \Rightarrow \neg\alpha)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

□

Although truth tables provide a valid proof for wffs of sentential logic, they are not applicable in a multi-variable setting where the truth value of a statement is dependent upon numerous changing values. Sentential logic fails to encompass such variable statements as well as verbiage like *there exists* and *for all*. Therefore it is necessary to expand our consideration to first order logic and its main component, the predicate.

2.6 Examples

At the end of each chapter, examples of concepts at work in the axiomatic systems of Zermelo-Fraenkel set theory, group theory, and Peano Arithmetic will be explored (though other instances of examples in the systems may occur elsewhere in the chapter as well).

Definition 2.55 An **axiomatic** system is an collection of wffs which are assumed to be true and from which the truth value of other wffs can be derived.

Such wffs with assumed but unproven truth values are called **axioms**, which comes from the greek 'axioma' meaning "to deem worthy" or "to require." Axioms, also commonly referred to as postulates or assumptions, cannot be proven from one another unless superfluous. Any true wff which can be obtained from the axioms via an accepted form of proof (e.g., truth tables and later tableaus) are called **thoerems**. The collection of all such theorems is is the **theory** of the system.

Sentential logic can be axiomatized, that is, be presented in the form of an axiomatic system, in numerous different ways, each still using (some of, but not neccessarily all) the symbols of the sentential logic language developed in this chapter. For instance, there is an axiomatization where wffs are defined as follows:

- Any proposition p, q, etc is a wff.
- If α and β are wffs, then $(\alpha \Rightarrow \beta)$ and $(\neg\alpha)$ are wffs.
- Nothing else is a wff

Any wff in one of the following forms is an axiom, where α, β, and γ are wffs:

1. $\alpha \Rightarrow (\beta \Rightarrow \alpha)$
2. $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$
3. $(\neg\beta \Rightarrow \neg\alpha) \Rightarrow ((\neg\beta \Rightarrow \alpha) \Rightarrow \beta)$

And there is a rule of inference, modus ponens where, if α and $(\alpha \Rightarrow \beta)$ are theorems, β is also a theorem.

Note that, although the axioms come in one of three forms (or"axiom schema"), there are an infinite number of them. A complete explanation of this axiomatic sentential logic could last another forty pages and there are many sources which can give an indepth analysis, so a complete explanation of these axioms will not be included here. However, an illustration of how theorems such as $\alpha \Rightarrow \alpha$ can be derived from the axioms using modus ponens will be provided:

1. $\alpha \Rightarrow ((\alpha \Rightarrow \alpha) \Rightarrow \alpha)$
   by axiom 1 where $\beta = (\alpha \Rightarrow \alpha)$.

2. $(\alpha \Rightarrow ((\alpha \Rightarrow \alpha) \Rightarrow \alpha)) \Rightarrow ((\alpha \Rightarrow (\alpha \Rightarrow \alpha)) \Rightarrow (\alpha \Rightarrow \alpha))$
   by axiom 2 where $\beta = (\alpha \Rightarrow \alpha)$ and $\gamma = \alpha$.

3. $(\alpha \Rightarrow (\alpha \Rightarrow \alpha)) \Rightarrow (\alpha \Rightarrow \alpha)$
   by 1, 2 and modus ponens.

4. $\alpha \Rightarrow (\alpha \Rightarrow \alpha)$
   by axiom 1 where $\beta = \alpha$.

5. $\alpha \Rightarrow \alpha$
   by 3, 4, and modus ponens.

It is also possible to move between the 'natural-deduction' system of sentential logic described in this chapter and this axiomatic one. If $(\alpha \wedge \beta)$ is defined to be $\neg(\alpha \Rightarrow \neg\beta)$, then $\wedge$ is understood to act as defined earlier in this chapter. This is especially evident as a truth table shows an equivalence between the two produces a tautology.

α	β	$\neg\beta$	$\alpha \Rightarrow \neg\beta$	$\neg(\alpha \Rightarrow \neg\beta)$	$\alpha \wedge \beta$	$(\neg(\alpha \Rightarrow \neg\beta)) \Leftrightarrow \alpha \wedge \beta$
T	T	F	F	T	T	T
T	F	T	T	F	F	T
F	T	F	T	F	F	T
F	F	T	T	F	F	T

There are numerous ways in which to express the sentential connectives: solely in terms of $\neg$ and $\Rightarrow$, or $\neg$ and $\wedge$, or $\neg$ and $\vee$, or even with the single operation $\alpha|\beta$, known as the Scheffer Stroke, which is equivalent to $\neg\alpha \vee \neg\beta$. A full description of the possible axiomatizations and various ways of defining connectives could fill its own thesis, but as sentential logic is merely a building block and not the focus, we will leave further study of these concepts up to the reader.

Zermelo-Fraenkel set theory, abbreviated ZF, is an axiomatic system which gives a theory for sets that avoid the paradoxes of naive set theory (recall example 2.5). It was the creation of German mathematician Ernst Friedrich Ferdinand Zermela and Israeli (though German-born) Abraham Halevi Fraenkel, the former which introduced it in 1908 and the latter which expounded upon it during the 1920s. ZF consists of eight axioms (though other sources may count differently). The addition of the famous, and originally controversial (in 1908), Axiom of Choice creates ZFC, Zermelo-Fraenkel set theory with the Axiom of Choice.

Group theory is the study of groups (sets with an associated binary operation) whose binary operations satisfy four axioms: closure, associativity, identity, and inverse. Group theory originates from three sources, including the theory of algebraic equations and number theory at the end of the 18th century, and geometry at the beginning of the 19th. In the theory of algebraic equations, mathematicians such as Joseph-Louis Lagrange and, later, Évariste Galois studied the permutations of the roots of equations, permutations which are now considered to be elements of a group. In number theory, Leonhard Euler explored the group properties of the remainders on division of powers a^n by a fixed prime p while Carl Gauss analyzed the group properties of the composition of equivalence classes of quadratics of the form $ax^2 + 2bxy + cy^2$.

Giuseppe Peano was a 19th century Italian mathematician who introduced a set of axioms he believed exemplified the natural numbers. A formal version of his postulates were published in 1889 in *The Principles of Arithmetic, Presented by a New Method.* These Peano axioms are often called the Dedekind-Peano axioms as they are actually a more strictly formulated version of those given by Richard Dedekind in 1888. These axioms, in conjunction with axioms defining the symbols $+$ and $\cdot$, form Peano Arithmetic, often abbreviated PA.

However, the axioms of these three systems, ZF, Group Theory, and PA, are expressed in terms of first order logic, using concepts which do not exist in sentential logic. Thus it is necessary to consider first order logic before these axioms can be given.

Chapter 3

First Order Logic

First order logic is an expansion of sentential logic which includes new symbols such as variables, parameters, predicates, and quantifiers. First order logic is also a two-valued logic, but rather than propositions being mapped to the set $\{\mathbf{T}, \mathbf{F}\}$, predicates have associated sets of ordered elements which can be mapped to $\{\mathbf{T},\mathbf{F}\}$ in the form of n-ary relations. The symbols of first order logic can be combined according to certain rules of formation to create wffs. The Unique Readability Theorem expands in this chapter to include first order logic and prove that every first order logic wff has a unique truth value when a specific ordered element is considered. It will be shown the truth values of these wffs are determined by the model in which they are evaluated.

3.1 An Introduction to First Order Logic

First order logic, or pure predicate logic, is a formal language with an infinite number of symbols that are classified as either logical or non-logical. The logical symbols include variables, connectives, parenthesis, and quantifiers while parameters, predicates, and functions are considered non-logical symbols. A verbal description will be given of these terms and later clarified using the idea of mapping.

Definition 3.1 A **variable** is a symbol which corresponds to a term, value, or individual which may change within the confines of a given statement. A **parameter** is a specific symbol that corresponds to a term, value, or individual which does not vary within a statement.

Thus a parameter is simply a set variable. Syntactically speaking, the only difference between the two is that while a variable may appear directly after a quantifier (defined later) in a wff, a parameter may not. It is standard form for mathematicians to use the first letters of the alphabet, such as a, b, and c to stand for constants, or parameters, while letters at the end, x, y, and z, indicate variables. Subscripts are often used to indicate (but not explicitely) a relation between symbols.

Example 3.2 Consider the parameters and variables of the following statements.

$\Diamond$ *Olympia is in Washington State.*
$\Diamond$ *Today is Christmas.*
$\Diamond$ $2 = 1$
$\Diamond$ $x = y$
$\Diamond$ $ax^2 + bx + c$

In the first sentence, both *Olympia* and *Washington State* are specific terms needed to understand the sentence and are thus parameters. In the second sentence, *Christmas* is also a parameter but *Today* is a variable as it changes depending on time and even location; on the other side of the globe 'today' takes a different value.

For the expression $2 = 1$, both 2 and 1 are parameters, but in $x = y$ both x and y are variables. The final line gives a general quadratic function. Here it is understood the unkown x is the variable but a, b, and c are parameters which determine which quadratic $f(x)$ is. □

Letters such as p, q, and r were used in sentential logic to indicate propositions. These characters are capitalized in first order logic to specify a predicate.

Definition 3.3 A **predicate** $P(x_1, x_2, \cdots, x_n)$ is a statement with n variables ($n \geq 0$) and a truth value when its variables are known. In other words, when a predicate's variables are replaced by parameters, it becomes a proposition.

Note that when $n = 0$ a predicate is a statement with no variables and a truth value, i.e., a proposition.

Example 3.4 Which of the following statments are predicates?

◊ x *is an even number.*
◊ $x + y = z$
◊ *The price of canned tuna is fifty cents*
◊ *Madagascar is an island.*

The first sentence is from example 2.2. There, the variable x prevented the sentence from being a proposition. However, when x is replaced by a parameter, the sentence does have a truth value and thus is a predicate. The second sentence has three variables x, y and z, while the third contains the variable *price of canned tuna.* When each of these variables are defined, each sentence has a truth value and thus each is a predicate. The final sentence is an example of a predicate with no variables, that is, a proposition. □

Note the truth values of these predicates, when their variables are determined, is also dependent upon the universe of discourse. Is $x + y = z$ being evaluated using standard arithmetic or modulo 4? Is *the price of canned tuna is fifty cents,* evaluated based on the pricing at Safeway or Walmart? This ambiguity will be addressed in the next section. Until then, the universe in which a predicate is considered will be the "obvious" one.

The x_i's of a predicate $P(x_1, x_2, \cdots, x_n)$ are understood to stand for the variables of the predicate. A predicate with three variables, such as $x + y = z$, is often symbolized by $P(x, y, z)$. If these variables become parameters a, b, and c, the former predicate (now propositions) will be denoted by $P(a, b, c)$. Similarly, a predicate may be written $P(x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_m)$ when m predicates have been substituted by parameters p_i or to simply indicate a dependence of the predicate on a parameter. For example, the predicate $P(x, y) : x + y = 0$ may alternatively be written as $P(x, y, 0)$. For convenience sake, a predicate is occasionaly written $P(x)$ (often when the number of variables is unknown) or P_n, where n is the number of variables.

Definition 3.5 The **arity** of a predicate $P(x_1, x_2, ..., x_n)$ is the number of variables under consideration.

Thus the arity of $P(x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_m)$ is n as every p_i is a paramter. Similarly, the arity of $P(x, y, z)$ is three, but the arity of $P(x, y, 0)$ is two.

Example 3.6 Consider the arity of the predicate $0 = \sum_{i=1}^{n} x_i$, where x_i is a real number.

$n = 1$: The 1-ary, or unary form of the predicate where $0 = x_1$
$n = 2$: The 2-ary, or binary, form of the predicate where $0 = x_1 + x_2$

$n = 3$: The 3-ary, or ternary, form of the predicate where $0 = x_1 + x_2 + x_3$

$\vdots$

$n = n$: The n-ary form of the predicate where $0 = x_1 + x_2 + ... + x_n$

$\square$

There is a joke that "escalators never break; they simply become stairs." The first order logic counterpart to this is that "predicates never have defined variables; they simply become propositions." Assigning values to a predicate's variables create a proposition, but a predicate also becomes a proposition by *quantifing* those variables.

Definition 3.7 The **existential quantifier**, denoted by the symbol $\exists$, gives the expression $\exists x P(x)$ a true truth value when there is at least one a which makes the predicate $P(a)$ a true proposition and a false truth value otherwise.

The existential quantifier $\exists$ is read as *there exists* or *for some.* For a variable x, $\exists x$ means there is at least one object x in the universe under consideration. When $\exists x P(x)$ is true, it is usually read as either *there exists x such that P(x)* or *for some x, P(x).*

Example 3.8 The following are examples of predicates that have had their variables quantified by the existential quantifier and are now propositions.

◊ *There exists a graduate student who has written a thesis*
about Gödel's Theorems.
◊ $\exists x \exists y (x = 2y)$.
◊ *There exists an even number which is odd.*

As long as one instance of the quantified variable is true, the entire proposition is true. In the above example, this implies the first two propositions are true (as the first one is verified by this thesis) while the last is false.

Realize though, that if the second predicate had been quantifed as $\exists x(x = 2y)$ it would remain a predicate.. □

The existential quantifier's counterpart in sententiall logic is the iterated conjunction. This means if x is an element of a potentially infinite set $\{a_1,\, a_2,\, \cdots,\, a_n,\, \cdots\ \}$, then $\exists x P(x)$ is equivalent to the multiple conjunction $P(a_1) \wedge P(a_2) \wedge \cdots \wedge P(a_n) \wedge \cdots$. This is especially evident when x can only take a finite number of values. For example, if x is one of the fifty states of the U.S. and $P(x_1, x_2, \cdots, x_n)$ is the proposition "*x is landlocked*", then $(\exists x P(x) \Leftrightarrow (P(Alabama) \vee P(Alaska) \vee ... \vee P(Wyoming))$. Alternatively stated, "*There exists a state which is landlocked*" is equivalent to "*Alabama or Alaska or ... or Wyoming is landlocked.*"

Definition 3.9 The **universal quantifier**, denoted by the symbol $\forall$, gives the expression $\forall x P(x)$ a true truth value when every a makes the predicate $P(a)$ a true propositions. It is false otherwise.

The universal quantifier is read as *for all* or *every.* For a variable x, $\forall x$ means for all objects x in the universe under consideration. When $\forall x P(x)$ is true, it is usually read as *for all x, P(x)* or *for every x, P(x).*

Example 3.10 The following are examples of predicates that have had their variables quantified by the universal quantifier and are now propositions.

◇ *Every graduate student has written about Gödel's Theorems.*
◇ $\forall x \forall y(x = 2y)$.
◇ *Every even number is divisible by 2.*

If even one quantified variable fails to be true, then the entire proposition is false. Thus the first two propositions are false and the last true. Again, if the second predicate had been quantified $\forall x(x = 2y)$, it would remain a predicate. □

The iterated disjunction is sentential logic's counterpart to the universal quantifier. Here, when x is an element of a potentially infinite set $\{a_1, a_2, \cdots, a_n, \cdots\}$, $\forall x P(x)$ is equivalent to $P(a_1) \wedge P(a_2) \wedge \cdots \wedge P(x_n) \wedge \cdots$. If x is month and $P(x)$ is the proposition "*x has at least 28 days*" then $(\forall x P(x) \Leftrightarrow (P(Jan) \wedge P(Feb) \wedge ... \wedge P(Dec))$. This can be also interpreted "*Every month has at least 28 days*" is equivalent to "*January and February and ... and December all have at least 28 days.*"

The connectives of sentential logic are integrated easily into first order logic, with the negation, conjunction, disjunction, implication, and equivalence of predicates rather than propositions. These connectives allow equivalences such as $(\forall x P(x) \Leftrightarrow \neg\exists x(\neg P(x))$ and $(\exists x P(x) \Leftrightarrow \neg\forall x(\neg P(x))$.

The quantification of variables also allows expressions such as DeMorgan's laws to translate into first order logic.

Example 3.11 In first order logic, DeMorgan's Laws are expressed using quantifiers rather than connectives. $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$ becomes $\neg\exists x P(x) \Leftrightarrow \forall x \neg P(x)$. Verbally, this says *"there is no x which makes P(x) true,"* is equivalent to "*P(x) is false for every x."* DeMorgan's other law $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$, becomes $\neg\forall x P(x) \Leftrightarrow \exists x \neg P(x)$, which says *"not all x make P(x) true,"* is equivalent to "*there is some x for which P(x) is false."*

A finite example helps illustrate DeMorgan's laws. Suppose $P(x)$ is a predicate and x is a variable which can only be replaced by a parameter from the set $\{a_1, a_2, a_3\}$. Then DeMorgan's laws state $\neg\exists x P(x) \Leftrightarrow \forall x \neg P(x)$ is

$$\neg(P(a_1) \vee P(a_2) \vee P(a_3)) \Leftrightarrow (\neg P(a_1) \wedge \neg P(a_2) \wedge \neg P(a_3))$$

and $\neg\forall x P(x) \Leftrightarrow \exists x \neg P(x)$ is

$$\neg(P(a_1) \wedge P(a_2) \wedge P(a_3)) \Leftrightarrow (\neg P(a_1) \vee \neg P(a_2) \vee \neg P(a_3)).$$

Example 3.12 The barber puzzle of example 2.6 can be stated in first order logic by using quantifiers. Recall that this puzzle states that there is a town with only one male barber who shaves only those men who do not shave themselves. Let x and y be variables that take the value of any man in town, $B(x)$ be the predicate which claims x is a barber, and $S(x, y)$ the predicate which states that x shaves y. Then the barber puzzle can be written

$$\exists x(B(x) \wedge \forall y(\neg S(y, y) \Leftrightarrow S(x, y))$$

The paradox is evident; the universal quantifier $\forall$ includes every y, man in town, including x, the barber. Thus when $x = y$, $\neg S(y, y) \Leftrightarrow S(y, y)$, an obvious impossibility. ☐

Quantifiers are often implied but not explicitly mentioned. The commutative property is often stated $a * b = b * a$ but is, in actuality,
$\forall a \forall b$, $a * b = b * a$. Other expressions, such as $x + x^2 = 0$ imply the existential quantifier and are written $\exists x$, $x + x^2 = 0$.

The propositions of sentential logic are described as simple and compound by their connective components. In first order logic, predicates are also labelled by the presence of quantifiers and connectives.

Definition 3.13 A **compound predicate** is a predicate with at least one connective or quantifier while a **simple predicate** is a predicate without them.

Example 3.14 Consider the following:

P(x): $x^2 + x = 0$
P(x, y, z): $\forall y((2x - z = 3y) \wedge (2z + y = 3x))$
P(n): $\exists n(n \in \mathbb{N})$

The first expression $P(x)$ is a simple predicate. The second sentence $P(x, y, z)$ is a compound predicate as it uses a conjunction and the universal quantifier. The third, $P(n)$, is a trick question; there is a existential quantifier, but that same quantifier makes $P(n)$ a proposition. However an altered $P(n)$ such as $P(m, n) : \exists n((m + n) \in \mathbb{N})$ is a compound predicate. □

The syllabary of first order logic thus includes the symbols of sentential logic and the addition of variables, parameters, predicates and quantifiers. Predicates do not have truth values as propositions do, but they can gain them when their variables are substituted for parameters or they are quantified. Therefore, to avoid any ambiguity surrounding the potential truth value of a predicate, the next section strictly define how predicates can be construed.

3.2 First Order Logic as a Relation

Predicates, like propositions, can be misinterpreted as English language sentences. Consider the statement *"Barack Obama is the president."* Is this a proposition or a predicate? "*Barack Obama*" seems to refer to a specific individual, but there may be other men who have the same name. Is "*president*" referring to an elected office of the United States or the person who holds it? As an office it is a parameter, as a person, a variable. Either way, there is not just a question of whether the sentence is a predicate, but also of the truth value it would have as a proposition.

Quantifiers can obscure the meaning of predicates which are English language sentences as well. Consider the predicates $P(x)$: "*x is perfect*" and $Q(y)$: "*y is nobody,*" where x and y are any person, i.e. x and y are 'someone.' Then $\neg\exists x P(x)$ is *"There does not exist a person who is perfect,"* or *"Nobody* *is perfect,"* either equally valid in english. Thus $\neg\exists x P(x) \wedge Q(y)$ is *"Nobody is perfect and someone is nobody."* The verbal wording draws the conclusion "*someone is perfect*," i.e. $P(y)$, which is patently impossible in first order logic, where this is $(\neg\exists x P(x) \wedge Q(y)) \Rightarrow P(y))$. Examination reveals the confusion arises from the use of 'nobody' as both a noun and a pronoun, the former meaning 'no person' and the latter 'a person of no importance.'

Thus new definitions and redefining old ones for first order logic is vital.

Definition 3.15 A **predicate** $P(x_1, x_2, ..., x_n)$ or P_n is a symbol.

A predicate is merely an element of the first order logic syllabary. When an n-ary relation is assigned to a predicate symbol, the relation creates an affiliation between the predicate's variables and the predicate resembles the description given by definition 3.3 (the truth value determined by this relation will be formalized at the end of the section).

Definition 3.16 An **n-ary relation** P_n^U on some non-empty set U is a subset $P_n^U \subseteq U^n$, $U^n = \{(x_1, x_2, ..., x_n) : x_1, x_2, ..., x_n \in U\}$.

A 1-ary relation P_1^U is also called a unary relation and is a subset of $U^1 = U$. A 2-ary, or binary, relation P_2^U is a subset of $U^2 = U \times U$ and a 3-ary, ternary, relation P_3^U is a subset of $U^3 = U \times U \times U$. The capital letter U is used because it indicates the 'universe' under discussion for an n-ary relation. The letter D is also often used because the subset corresponds to the 'domain' of discourse. The elements of this universe are the parameters described in the previous section. The universe U can be varied, from the natural numbers to vector spaces to geography. The universe can be finite or infinite, countable or uncountable. Examples 3.17 through 3.22 give various n-ary relations on the specified universe.

Example 3.17 The simplest interpretation of an n-ary relation is as some set. If $U = \mathbb{R}$, the following are n-ary relations on U.

$P_1^{\mathbb{R}} = \{6, \pi, 10004, \sqrt{2}\}$
$Q_1^{\mathbb{R}} = \{..., -4, -2, 0, 2, 4, ...\}$
$R_2^{\mathbb{R}} = \{(3, 0)\}$
$S_3^{\mathbb{R}} = \{(1, 2, -3), (\pi, 2, -3), (1, 2, \frac{1}{3}), (\pi, 2, \frac{1}{3})\}$
$T_n^{\mathbb{R}} = \emptyset$ (Any n-ary relation may simply be the empty set)

□

Often such n-ary relations are written in set builder notation $\{(x_1, x_2, ..., x_n) \in U^n : \textit{condition}\}$ to avoid listing a potentially infinite number of elements of U^n. In this manner, $Q_1^{\mathbb{R}}$ can be written as $\{x \in \mathbb{R} : x \textit{ is even}\}$. Similarly, in this notation, any parameters given by the relation can be expressed. In $S_3^{\mathbb{R}}$, 2 is parameter (it remains fixed), thus $S_3^{\mathbb{R}} = \{(x, 2, y) : x \in \{1, \pi\} \wedge y \in \{-3, \frac{1}{3}\}\}$. However, note that $S_3^{\mathbb{R}}$ corresponds to a ternary predicate $S_3 = S(x, y, z)$ rather than a binary predicate S(x, 2, z).

Example 3.18 Let $U = \mathbb{Z}$. The following are n-ary relations on U.

$P_1^{\mathbb{Z}} = \{x \in \mathbb{Z} : x > 0\}$
$Q_2^{\mathbb{Z}} = \{(x, y) \in \mathbb{Z}^2 : x + y \textit{ is odd }\}$
$R_3^{\mathbb{Z}} = \{(x, y, z) \in \mathbb{Z}^3 : x < y < z\}$

□

Example 3.19 Let $U = \mathbb{C}$. The following are n-ary relations on U.

$P_1^{\mathbb{C}} = \{x \in \mathbb{C} : ||x|| = 0\}$
$Q_2^{\mathbb{C}} = \{(x, y) \in \mathbb{C}^2 : x \textit{ and } y \textit{ are real }\}$
$R_3^{\mathbb{C}} = \{(x, y, i) \in \mathbb{C}^3 : x + y = i\}$

□

Example 3.20 Let $U = \{\textit{the states of the United States of America}\}$. The following are n-ary relations on U.

$P_1^U = \{x \in U : x \textit{ is an island}\}$
$Q_2^U = \{(x,y) \in U^2 : x \textit{ has more land than } y\}$
$R_3^U = \{(x,y,z) \in U^3 \; x \textit{ borders both } y \textit{ and } z\}$

□

Example 3.21 Let U be the constructed formal languge of example 2.24. The following are n-ary relations on U.

$P_1^U = \{x \in U : x \textit{ begins with ‘c’}\}$
$P_2^U = \{(x,y) \in U^2 : x \textit{ and } y \textit{ contain the same symbols}\}$
$P_3^U = \{(x,y,z) \in U^3\text{:} \; x\textit{, } y\textit{, and } z \textit{ contain only one symbol in common}\}$

□

Example 3.22 Let $U = \mathbb{R}$. The following are n-ary relations on U

$$
\begin{aligned}
&P_1^{\mathbb{R}} = \{x_1 \in \mathbb{R} : 0 = x_1\}\\
&P_2^{\mathbb{R}} = \{(x_1, x_2) \in \mathbb{R}^2 : 0 = x_1 + x_2\}\\
&P_3^{\mathbb{R}} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 = x_1 + x_2 + x_3\}\\
&\vdots\\
&P_n^{\mathbb{R}} = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : 0 = x_1 + x_2 + ... + x_n\}
\end{aligned}
$$

□

This example may seem familiar; it is from example 3.6 where the arity of the predicate $0 = \sum_{i=1}^{n} x_i$ was being discussed. This is because every predicate given in that previous example had the above implied n-ary relation assigned to it. N-ary relations create relationships between the n variables of a predicate that can (but not necessarily should) be expressed as an english language sentence. The following two examples show how various n-ary relations on a set U can be assigned to predicates (such assignments will be addressed again in section 3.6).

Example 3.23 Let $U = \{2,3\}$ Then the unary relations are $\{2\}$, $\{3\}$, $\{2,3\}$, and $\emptyset$. Below are examples of predicates in U, each assigned one of the possible unary relations, and statements which could express the relation.

$P_1 \to P_1^U = \{2\}$: *The set of even numbers from* U, $\{x \in U : x \neq 3\}$
$Q_1 \to Q_1^U = \{3\}$: $\{x \in U : x \textit{ is odd}\}$
$R_1 \to R_1^U = \{2,3\}$: *The set of prime numbers in* U, $\{x \in U\}$
$S_1 \to S_1^U = \emptyset$: *The set of imaginary numbers in* U

Notation: The number 1 can be dropped on a unary relations P_1^U as it is with predicates with only one variable, i.e., $P_1 \to P_1^U$ is often just $P \to P^U$.

Binary predicates can be assigned binary relations as well. (Note that unlike above, not all possible relations are listed).

$P_2 \to P_2^U = \{(2,2)\}$: *The size and the lowest number of* U
$Q_2 \to Q_2^U = \{(2,2),(3,3)\} : \{(x,y) \in U^2 : x = y\}$
$R_2 \to R_2^U = \{(2,2),(3,2)\} : \{(x,2) \in U^2\}$
$S_2 \to S_2^U = \{(2,2),(2,3),(3,2),(3,3)\} : \{(x,y) \in U^2 : (x+y) \in \{4,5,6\}\}$

□

N-ary relations allow n-ary predicates to have clear and non-subjective truth values. If $P_n = P(x_1, x_2, \cdots, x_n)$ is a predicate in a universe U with a n-ary assignment P_n^U, then $P(a_1, a_2, \cdots, a_n)$ is **T** if $(a_1, a_2, \cdots, a_n) \in P_n^U$ and **F** if $(a_1, a_2, \cdots, a_n) \notin P_n^U$, where every $(a_1, a_2, \cdots, a_n) \in U^n$. Thus if $P_n^U = \emptyset$, $P(a_1, a_2, \cdots, a_n)$ is false for every ordered element $(a_1, a_2, \cdots, a_n)$ as the empty set contains no elements. Similarly, if $P_n^U = U^n$, $P(a_1, a_2, \cdots, a_n)$ is true for every ordered element $(a_1, a_2, \cdots, a_n)$, as U^n contains every possible element.

Although none of the prior examples show an 0-ary relation, they do exist. As $P_0^U \subseteq U^0 = \{()\}$ where U^0 is the set containing the empty string, P_0^U

can only be of two forms; the set containing the empty string, $\{()\}$, and the empty set, $\emptyset$. Thus if $P_0^U = \emptyset$, $P_0 = P()$ is false and if $P_0^U = \{()\}$, $P_0 = P()$ is true. This natural mapping from $\{()\}$ to the truth value **T** and from $\emptyset$ to the truth value **F** is reasonable as P_0^U is assigned to the predicate P_0 which has no variables and thus is a proposition. This means every proposition can be written as a 0-ary predicate P_0 whose truth value is determined by the 0-ary relation P_0^U.

Now that the truth value of a simple predicate is non-subjective, it is time to consider the ways predicates can be combined and still have meaning; that is, we must turn our attention to the syntax of first order logic and the well-formed formula it creates.

3.3 Syntax

This chapter will focus on developing a formal grammer for first order logic and showing how a parsing sequence proves whether or not a string of first order logic is a wff. Parsing sequences will lay the stage for proving the truth values of these wffs in later sections.

Definition 3.24 In first order logic a **well-formed formula**, or **wff**, is a string which follows the syntax of first order logic, i.e. can be obtained by applying a finite number of times the following rules of formation:

p: If p is a proposition in sentential logic, then p is a wff.
P_n: If P_n is an n-ary predicate then P_n is a wff.
$\neg$: If α is a wff, then $\neg\alpha$ is a wff.
$\wedge$: If α and β are wffs, then $(\alpha \wedge \beta)$ is a wff.
$\vee$: If α and β are wffs, then $(\alpha \vee \beta)$ is a wff.
$\Rightarrow$: If α and β are wffs, then $(\alpha \Rightarrow \beta)$ is a wff.

$\Leftrightarrow$: If α and β are wffs, then $(\alpha \Leftrightarrow \beta)$ is a wff.
$\forall$: If α is a wff and x is a variable, then $\forall x \alpha$ is a wff.
$\exists$: If α is a wff and x is a variable, then $\exists x \alpha$ is a wff.

Definition 3.25 A **compound wff** is a wff which uses at least one connective or quantifier while a **simple wff** is a wff that does not use any.

Example 3.26 Consider the following:

1: $P(x, y)$
2: $\exists x P(x, y)$
2: $\exists x \forall y P(x, y)$

The first expression, $P(x, y)$, is a simple wff while the second $\exists x P(x, y)$ is a compound wff. The third, although all of its variables have been quantified, is also a compound wff (the quantified variables creates a proposition). □

Example 3.27 Find a parsing sequence for the string $(\exists x P(x, y) \Rightarrow (\forall y P(x, y) \Leftrightarrow q))$, where $P(x, y)$ is a predicate and q is a proposition.

1: $P(x, y)$ is a predicate by hypothesis and thus a wff by P_n.
2: q a proposition by hypothesis and thus a wff.
3: $\exists x P(x, y)$ is a wff by $\exists$ and 1.
4: $\forall y P(x, y)$ is a wff by by $\forall$ and 1.
5: $(\forall y P(x, y) \Leftrightarrow q)$ is a wff by $\Leftrightarrow$, 4, and 2.
6: $(\exists x P(x, y) \Rightarrow (\forall y P(x, y) \Leftrightarrow q))$ is wff by $\Rightarrow$, 3, and 5.

Thus the string is a wff. □

Example 3.28 Consider the parsing sequence for the string $\exists x(P(x, y) \Rightarrow \forall y(P(x, y) \Leftrightarrow q))$, where $P(x, y)$ is a predicate and q a proposition.

1: $P(x, y)$ is a predicate by hypothesis and thus a wff by P_n.
2: q a proposition by hypothesis and thus a wff$p$.
3: $(P(x, y) \Leftrightarrow q)$ is a wff by $\Leftrightarrow$, 2, and 1.
4: $\forall y(P(x, y) \Leftrightarrow q)$ is a wff by $\forall$ and 3.
5: $(P(x, y) \Rightarrow \forall y(P(x, y) \Leftrightarrow q)$ is a wff by $\Rightarrow$, 1 and 4
6: $\exists x(P(x, y) \Rightarrow \forall y(P(x, y) \Leftrightarrow q))$ is a wff by $\exists$.

Thus the string is a wff. $\square$

Although these strings differ in only the location of a few parenthesis, their parsing sequences have different orders. While the main connnective of example 3.27 was $\Rightarrow$, the main connective of was $\exists$. This is indicative of the importance of tracking parenthesis, especially around quantifiers and when working with abbreviated wffs.

When evaluating abbreviated wffs, the quantifiers $\exists$ and $\forall$ have the same importance as $\neg$ in the order of operations. When abbreviating a wff which contains $\exists x\alpha$ or $\forall x\alpha$, often parenthesis are added, rather than taken away, to prevent confusion. For example, the abbreviated wff $\exists xP(x) \Leftrightarrow Q(y)$ is $(\exists xP(x) \Leftrightarrow Q(y))$ rather than $\exists x(P(x) \vee Q(y))$. But these can easily be confused, so often the abbreviation is written $(\exists xP(x)) \Leftrightarrow Q(y)$, with the 'extra' parenthesis.

3.4 Uniqueness of Truth Values

Before truth values can be assigned for wffs of first order logic, it is necessary to prove every wff can only be interpreted one way. Although this was shown for wffs of sentential logic, wffs of first order logic also include predicates and quantifiers, so it is necessary to reprove the theorems of the previous chapter.

Theorem 3.29 *Every wff is balanced.*

Proof: Recall that a wff α is balanced if it has the same number of left brackets as right brackets (definition 2.33). If α has lenth 0 it is the empty string and hence balanced. If the length is 1, then α is either a proposition or a predicate, both of which have no brackets and thus are balanced. Assume every wff of length n or less is balanced and suppose α is a wff of length $n+1$. Then either $\alpha = (\neg\beta)$, $\alpha = (\beta_1 * \beta_2)$ where $*$ is a binary connective, or $\alpha = \forall x\beta$ or $\alpha = \exists x\beta$ where β, β_1 and β_2 are wffs.

If α is of the first two forms, it is balanced by the reasoning of theorem 2.34. If α begins with a quantifier, β has length of less than n and is balanced. α would not have any more parenthesis than β in this case, so α would also be balanced. Thus, by induction, every wff of first order logic is balanced. ▮

Theorem 3.30 *Any string which is an initial part of a wff is not a wff.*

Proof: If α has a length of 0 then it is an empty string and has no initial part. If the length is 1, α must be either a predicate or a proposition and would have no initial part. Assume that any initial part of a wff with length at most n is not a wff and let α be a wff of length n+1. Then $\alpha = (\neg\beta)$, $\alpha = (\beta_1 * \beta_2)$ where $*$ is a binary connective, $\alpha = \forall x\beta$, or $\alpha = \exists x\beta$, where β, β_1 and β_2 are wffs. By the reasoning in theorem 2.37, no initial string of the first two cases can be a wff, so it remains to consider the final cases.

Let s be an initial part of the wff $\alpha = \forall x\beta$ and assume s is a wff. Then $\alpha = st$ where t is not the empty string. s begins with a quantifier and is a wff, so cannot be simply the strings $\forall$ or $\forall x$, but must be of the form $\forall x s_1$ where s_1 is a wff. Then $\forall x\beta = \forall x s_1 t$ and removal of the first two symbols yields, $\beta = s_1 t$. However, both of the wffs β and $s_1 t$ have a length of less than n and thus cannot be initial parts of each other by the induction hypothesis. Hence, as they begin at the same place within α, $\beta = s_1 t$. But this implies t is the empty string and s is therfore not an initial part of the wff, a contradiction.

The reasoning for $\alpha = \exists x\beta$ is the same, thus any string which is an initial part of a wff is not a wff in first order logic. ▮

Theorem 3.31 The Unique Readability Theorem: *Every wff α which begins with a left parenthesis has exactly one main connective.*

Proof: This proof is identical to that given for sentential logic Unique Readability Theorem, except that the wffs are now of first order logic. The similarity of the proofs is reasonable as the first order logic wffs which are formed from predicates or by quantifiers do not have parenthesis and thus are not essential to the theorem. ▮

Thus it is known every wff is uniquely interpreted and so every parsing sequence of a wff gives the same truth value. However, before those truth values can be determined, it is necessary to expound on the types of variables which can occur in wffs.

3.5 Variables

Variable are essential to predicates; without them they are propositions. A specific variable can occur numerous times in the same wff and in two different capacities, “free” and “bound.” Each place in a wff where a symbol or string is located is referred to as an occurence.

Definition 3.32 A **bound variable** or **dummy variable** x is a variable on which an expression does not depend, while a **free variable** x is a variable which is not bound.

Free variables are those which, when replaced by parameters, determine the truth value of a wff.

Example 3.33 In the following, y is a free variable.

◊ $P(y)$
◊ $\forall x P(x, y)$
◊ $\int_{-\infty}^{\infty} x^y dx$

□

Bound variables, on the other hand, can be replaced everywhere by another variable y (where y does not occur elsewhere in the wff) without affecting the potential truth value of a wff.

Example 3.34 In the following, x is a bound variable:

◊ $\exists x P(x)$
◊ $\forall x P(x)$
◊ $\exists x$ such that $0 = 1 + x + x^2$
◊ $\neg \exists x p(x) \Leftrightarrow \forall x \neg p(x)$ (DeMorgan's Law)
◊ $\forall x f(x, y)$

Note that in DeMorgan's Law, x actually stand for two different bound variables. It is possible to separately substitute y and z for each bound instance of x to obtain the (equivalent) $\neg \exists y p(y) \Leftrightarrow \forall z \neg p(z)$. □

The action of replacing a bound variable x with another variable y in a wff α is called a substitution of variables. However, problems arise when the substitution of y causes a bound occurence when there should be a free one. For example, consider the true wff $\alpha : \forall x \exists y(x + y > 0)$. Then when y replaces x this becomes $\forall y \exists y(y + y > 0)$ or $\forall y(y + y > 0)$ which is false. A substitution which avoids this is referred to as a free substitution of variables and is denoted by $\alpha(x//y)$. This notation can also indicate the replacement of a variable by a parameter, (x//a). When every variable $(x_1, x_2, \cdots, x_n)$ is replaced by parameters $(a_1, a_2, \cdots, a_n)$, $\alpha((x_1, x_2, \cdots, x_n//a_1, a_2, \cdots, a_n)$ is called an instance of α.

Definition 3.35 Let α be a wff which begins with a quantifier. The **scope** of that quantifier is α.

This means α is of the form $Qx\beta$ where Q is a quantifier, x a variable, and β a wff. The scope of Q includes itself, x, and β.

Example 3.36 Consider the wff $\exists x(\forall y P(x, y, z) \Leftrightarrow \forall z Q(x, y, z))$. The scope of the existential quantifier is the entire wff, including $\exists x$. The scope of the first universal quantifier is $\forall y P(x, y, z)$ while the scope of the second is $\forall z Q(x, y, z)$.

□

The definitions of free and bound variables expand to include this notion of scope. A bound variable occurs only in the scope of a quantifier which has quantified it. A free variable is any variable which doesn't.

Example 3.37 Consider the free and bound variables of the wff
α: $\exists x(\forall y P(x, y, z) \Leftrightarrow \forall z Q(x, y, z))$ introduced in example 3.36. Every occurence of x is bound, but while the first two occurrences of y in $\forall y P(x, y, z)$ are bound, the occurence of y in $Q(x, y, z)$ is free. Conversely, the first occurence of z in $P(x, y, z)$ is free, but the last two in $\forall z Q(x, y, z)$ are bound.

Note that if the bound occurences of the variables x, y, and z were replaced everywhere by some other variables such as u, v, and w, α retains the same meaning : $\exists u(\forall v P(u, v, z) \Leftrightarrow \forall w Q(u, y, w))$ □

For the above example, each quantifier had only a single scope. But is this always the case or is it possible for there to be multiple scopes for the same quantifier or even no scope at all?

Theorem 3.38 The Unique Scope Theorem*: There is a unique scope for every occurence of a quantifier in a wff.*

Proof: It is necessary first to prove the existence of a scope for a quantifier before proving its uniqueness. Let α be a wff and $\mathcal{Q}$ a quantifier which appears in α. As α is a wff, by the rules of formation, when $\mathcal{Q}$ occurs, it must be in the form $\mathcal{Q}x\beta$ where x is a variable and β a wff. But $\mathcal{Q}x\beta$ is a wff by the rules of formation, so $\mathcal{Q}$ has the scope $\mathcal{Q}x\beta$.

For uniqueness of this scope, let $\mathcal{Q}$ again be a specific quantifier which appears in the wff α and suppose β_1 and β_2 are both wffs in α that begin with $\mathcal{Q}$. As β_1 and β_2 are both wffs which begin at $\mathcal{Q}$, by theorem 3.30, neither can be an initial part of the other, so β_1 and β_2 must be the same. Thus there is only one scope for $\mathcal{Q}$. ▮

Given a n-ary relation on a predicate in a universe U, it is necessary to know whether a variable is free or bound in that predicate to determine the truth value of it. Such an assignment of an n-ary relation to a predicate is referred to as a model, and is explored in the next section.

3.6 Models

This section will focus on models and the criteria they provide to determine the truth values of wffs of any universe U. Such wffs are written using (but not limited to) connectives, quantifiers, predicates, parameters, and variables. These symbols form the **language** $\mathcal{L}$ of U. $\mathcal{L}$ will be explored further in the next section, though note that a model assigns meaning to those symbols of $\mathcal{L}$.

Definition 3.39 A **model** M of a non-empty set U (called the universe of the model) is a system where every predicate P_n (and proposition P_0) is assigned an n-ary relation P_n^U.

Example 3.40 Consider example 3.23 where $U = \{2, 3\}$. If there are two unary predicates P and Q and one binary predicate R_2, then the unary relations $P^U = \{2\}$ and $Q^U = \{3\}$ and binary relation $R_2^U = \{(2, 3)\}$ make a model. Any other combination of relations (such as $P^U = \{3\}$, $Q^U = \emptyset$, and $R_2^U = \{(2, 2), (3, 3)\}$) gives another model. $\square$

Definition 3.41 To find the truth value α_M of a wff α in a model M, simply use the following rules:

P_n: If α is a predicate $P(a_1, a_2, \cdots, a_n)$, then $\alpha_M = \mathbf{T}$ iff
$(a_1, a_2, \cdots, a_n) \in P_n^U$

$\neg$: If $\alpha_M = \mathbf{T}$ then $(\neg\alpha)_M = \mathbf{F}$.
If $\alpha_M = \mathbf{F}$ then $(\neg\alpha)_M = \mathbf{T}$.

$\wedge$: If $\alpha_M = \mathbf{T}$ and $\beta_M = \mathbf{T}$ then $(\alpha \wedge \beta)_M = \mathbf{T}$.
Otherwise, $(\alpha \wedge \beta)_M = \mathbf{F}$.

$\vee$: If $\alpha_M = \mathbf{F}$ and $\beta_M = \mathbf{F}$ then $(\alpha \vee \beta)_M = \mathbf{F}$.
Otherwise, $(\alpha \vee \beta)_M = \mathbf{T}$.

$\Rightarrow$: If $\alpha_M = \mathbf{T}$ and $\beta_M = \mathbf{F}$ then $(\alpha \Rightarrow \beta)_M = \mathbf{F}$.
Otherwise, $(\alpha \Rightarrow \beta)_M = \mathbf{T}$.

$\Leftrightarrow$: If $\alpha_M = \beta_M$ then $(\alpha \Leftrightarrow \beta)_M = \mathbf{T}$.
If $\alpha_M \neq \beta_M$ then $(\alpha \Leftrightarrow \beta)_M = \mathbf{F}$.

$\forall$: $(\forall x\alpha)_M = \mathbf{T}$ iff $\alpha(x//a)_M$ for every $a \in U$.

$\exists$: $(\exists x\alpha)_M = \mathbf{T}$ iff $\alpha(x//a)_M$ for at least one $a \in U$.

(Recall $\alpha(x//a)$ indicates every free variable x in α is being replaced by the parameter a from U.)

Example 3.42 Consider the wff $(\exists x P(x) \wedge \forall y Q(y))$ in the model M where $U = \{2, 3\}$, $P(x)$ and $Q(y)$ are unary predicates, and their unary relations are $P^U = \{2\}$ and $Q^U = \{3\}$. To find the truth value of this wff, it is necessary to find its parsing sequence and evaluate the truth value at each step.

1: $P(2) = \mathbf{T}$ and $P(3) = \mathbf{F}$ by P^U of the hypothesis and P_n of the truth value rules.
2: $Q(3) = \mathbf{T}$ and $Q(2) = \mathbf{F}$ by Q^U and P_n
3: $\exists x P(x) = \mathbf{T}$ by 1 and $\exists$.
4: $\forall y Q(y) = \mathbf{F}$ by 2 and $\forall$.
5: $(\exists x P(x) \wedge \forall y Q(y)) = \mathbf{F}$ by 3, 4, and $\wedge$.

Thus the wff is false in this model.

□

Example 3.43 Consider the wff $\exists x \forall y R(x, y)$ in the model M where $U = \{2, 3\}$, $R(x, y)$ is a binary predicate, and its binary relation is $R_2^U = \{(2, 2), (2, 3), (3, 2)\}$.

1: $R(2, 2)$, $R(2, 3)$, $R(3, 2)$ are $\mathbf{T}$. $R(3, 3)$ is $\mathbf{F}$.
2: $\forall y R(x, y)$: $R(x, 2) = \mathbf{T}$ and $R(x, 3) = \mathbf{T}$ iff $x = 2$.
3: $\exists x \forall y R(x, y) = \mathbf{T}$ by 1 and 2.

Hence the wff is true in this model.

□

When a wff α is true in a model M, M is said to satisfy, or model, the wff and is denoted by $M \models \alpha$ or $\alpha_M = \mathbf{T}$. If α is false in a model, then $M \not\models \alpha$ or $\alpha_M = \mathbf{F}$. Similary, M can model a set of wffs H, denoted $M \models H$ when M models every wff in H. The rules for a wff satisfying a model M in first order logic can be extended from those of sentential logic given in proposition 2.48.

Proposition 3.44 If M is a model with a universe U, α is a wff, x is a variable, and a is an element of U, then:

$\exists$: If $M \models \exists x \alpha$ then $M \models \alpha(x//a)$ for some a.
$\neg\exists$: If $M \models \neg\exists x \alpha$ then $M \models \neg\alpha(x//a)$ for every a.
$\forall$: If $M \models \forall x \alpha$ then $M \models \alpha(x//a)$ for every a..
$\neg\forall$: If $M \models \neg\forall x \alpha$ then $M \models \neg\alpha(x//a)$ for some a.

Proof: The reasoning for this proposition follows from the notion of M modeling a wff α and the definition of the truth value α_M of a wff α in a model M. For example, $M \models \exists x \alpha$ means $\alpha(x//a)_M = \mathbf{T}$ for at least one $a \in U$, so $M \models \alpha(x//a)$. Likewise, if $M \models \neg\exists x \alpha$ then $\neg\alpha(x//a)_M = \mathbf{T}$ for every $a \in U$ so $M \models \neg\alpha(x//a)$. The reasoning for the rest is similar, and so is left to the reader. ∎

The tautology of sentential logic has an analogy in first order logic: the valid wff.

Definition 3.45 A wff α is **valid** if it holds (is true) in every model of its universe U.

A few very small examples help illustrate this concept.

Example 3.46 Let $U = \{0, 1\}$ and there be only one unary predicate $P(x) = P$. There are four possible models, given by the different unary relations $P^U = \emptyset$, $P^U = \{0\}$, $P^U = \{1\}$, and $P^U = \{0, 1\}$. Coinsider the wff $\forall x P(x) \Rightarrow \exists x P(x)$. It is easy to see in the first three models the antecedent will be false and thus the entire wff will be true. For the last, $P(x)$ is true for every element, so both $\forall x P(x)$ and $\exists x P(x)$ are true, and thus the entire wff is true. Hence $\forall x P(x) \Rightarrow \exists x P(x)$ is valid in the universe U. $\square$

Example 3.47 Let $U = \{0, 1\}$ as before, but let there be one binary predicate $P(x, y) = P_2$. Then there are sixteen possible models:

1: $P_2^U = \emptyset$
2: $P_2^U = \{(0, 0)\}$
3: $P_2^U = \{(0, 1)\}$
4: $P_2^U = \{(1, 0)\}$
5: $P_2^U = \{(1, 1)\}$
6: $P_2^U = \{(0, 0), (0, 1)\}$
7: $P_2^U = \{(0, 0), (1, 0)\}$
8: $P_2^U = \{(0, 0), (1, 1)\}$
9: $P_2^U = \{(0, 1), (1, 0)\}$
10: $P_2^U = \{(0, 1), (1, 1)\}$
11: $P_2^U = \{(1, 0), (1, 1)\}$
12: $P_2^U = \{(0, 0), (0, 1), (1, 0)\}$
13: $P_2^U = \{(0, 0), (0, 1), (1, 1)\}$
14: $P_2^U = \{(0, 0), (1, 0), (1, 1)\}$
15: $P_2^U = \{(0, 1), (1, 0), (1, 1)\}$
16: $P_2^U = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$

Consider the wff $\exists x \forall y P(x,y) \Rightarrow \forall y \exists x P(x,y)$. For the antecedent to be false, P^U_x must lack at least one element (x,y) where $y = 1$ or $y = 2$. Thus, the antecedent is false in models 1-5, 7, and 10, so the wff is true in these models. For the consequent to be true, there must be at least two elements of U^2 for which $P(x,y)$ is true, one in which $y = 0$ and one in which $y = 1$. Thus the consequent is true for models 6, 8, 9, and 11-16 and so is the wff. Hence the wff holds in every model and thus is valid. $\square$

Unfortunately, most wffs of first order logic cannot be proven valid via case analysis as the previous two were. The wff $\forall x P(x) \Rightarrow \exists x P(x)$ of example 3.46 is actually valid for any non-empty set U, but that is much more difficult proof. As the number of predicates increases, the number of models grows even faster. For example, if M is a model in first order logic with a universe U that contains n elements and only one unary predicate P^U, there are 2^n number of models (In example 3.46, $n = 2$ and there were 2^2 models). Two unary predicates increases this to $(2^n)^2$ possible models. A single binary predicate gives (2^{n^2}) models (in example 3.47, $n = 2$ and there were $(2^{2^2} = 16$ models). An infinite number of predicates, which is often the case, gives an infinite number of possible models to consider. To prove the validity of a first order logic wff for all models and for universes of infinite size, it is necessary to use a proof which apply in such settings: the semantic tableau.

3.7 Examples

With first order logic it is now possible to explore the postulates of the axiomatic systems of Zermelo-Fraenkel set theory, group theory, and Peano Arithmetic. The axioms of each system define the elements of the universe U

in which we will consider various model. There is also a language $\mathcal{L}$ that will be taken into consideration for each system. Certain symbols and their associated meanings, such as connectives, quantifiers, variables, and parenthesis will be implied rather than explicitely listed in $\mathcal{L}$. Similarly, the equality symbol '=' will be a member of any language where equality is axiomatically defined as follows:

Axioms of Equality (EA):
EA1: $\forall x(x = x)$ *Equality is reflexive*
EA2: $\forall x \forall y(x = y \Rightarrow y = x)$ *Equality is symmetric*
EA3: $\forall x \forall y \forall z(x = y \wedge y = z \Rightarrow x = z)$ *Equality is transitive*
EA4: $\forall a \forall b, (a \in H \wedge a = b \Rightarrow b \in H)$ *Sets are closed under equality*

These four postulates are known as the Equality Axioms and explicitely define the symbol =. A model M adheres to these axioms if for every a and b in a universe U, $M \models a = b$ if and only if a is quantitatively the same as b, that is, if for a predicate $P(x)$, $(P(a) \Leftrightarrow P(b)) \Rightarrow a = b$. The axioms ensure the reflexivity, symmetry, transitivity, and closure of equality. Thus when a proof cites EA4, it uses the specific axiom implied by the existence of the equality symbol in the system under consideration. For example, in ZF, two sets are equal if they contain the same elements, which is the first of eight axioms of Zermelo-Fraenkel set theory.

Axioms of Zermelo-Fraenkel set theory (ZF):
ZF1: $\forall z(z \in X \Leftrightarrow z \in Y) \Rightarrow X = Y$ *Axiom of Extensionality*
ZF2: $\forall x \forall y \exists Z(x \in Z \wedge y \in Z)$ *Axiom of Pairing*
ZF3: $\forall X \forall \alpha \exists Y \forall y \big(y \in Y \Leftrightarrow \big(y \in X \wedge \alpha(y)\big)\big)$ *Axiom of Subsets*
ZF4: $\forall X \exists Y \forall x \big(x \in Y \Leftrightarrow \exists z(z \in X \wedge x \in z)\big)$ *Axiom of Union*
ZF5: $\forall X \exists Y \forall x(x \in Y \Leftrightarrow x \subseteq X)$ *Axiom of the Power Set*
ZF6: $\exists X \big(\emptyset \in X \wedge \forall x \big((x \in X) \wedge (x \cup \{x\} \in X)\big)\big)$ *Axiom of Infinity*
ZF7: $\forall x \forall y \forall z \big(\beta(x, y) \wedge \beta(x, z) \Rightarrow y = z \Rightarrow$
$\forall X \exists Y \forall y \big(y \in Y \Leftrightarrow (\exists x \in X)\beta(x, y)\big)\big)$ *Axiom of Replacement*

ZF8: $\forall X\big(x \neq \emptyset \Rightarrow \exists x\big((x \in X) \wedge (X \cap x = \emptyset)\big)\big)$ *Axiom of Foundation*

(where $\emptyset = \{y : y \neq y\}$)

Zermelo-Fraenkel set theory is a universe which consists of various sets (including $\emptyset$) which comply with the above axioms. The language of ZF includes symbols of set thoery such as $\in$, $\cup$, $\cap$, $\subseteq$, etc (as well as the previously mentioned sets).

The first postulate, the Axiom of Extensionality, simply states that if two sets have the same elements they are equal. The Axiom of Pairing is also known as the Axiom of the Unordered Pair and establishes that for any two elements (potentially sets) there exists a set which contains only those two elements. The Axiom of Union is also refrerred to as the Axiom of the Sum Set and gives that for any set there is another set which is the union of all elements of the first set. The Axiom of the Power Set simply says that for any set X there is a set Y=P(X) which is the set of all subsets of X, referred to as the power set. The Axiom of Infinity ensures the existence of an infinite set. The Axiom of Foundation is also known as the Axiom of Regularity and states that every non-empty set contains a subset which is disjoint, that is, they have no elements in common.

Before discussing the Axiom of Subsets and the Axiom of Replacement it is important to note both incorporate, respectively, a unary predicate α and a binary predicate β. Recall that an n-ary relation such as α_1^{ZF} or β_2^{ZF} is a set, thus when predicates $\alpha(x)$ or $\beta(x, y)$ are true it is equivalent to stating $x \in \alpha_1^U$ or $(x, y) \in \beta_2^U$.

Other names for the Axiom of Subsets include (but is not limited to) the Axiom of Separation, Axiom of Comprehension, and Axiom of Specification. The Axiom of Subsets states that given a property described by a predicate

α, then, for any set X there is a set Y that contains every element x of X which has that property, that is, for which $\alpha(x)$ is true. Note that this means a set S cannot be defined as $S = \{s : s \notin s\}$ as each individual s must belong to a specified "Y." Thus there is a restriction to X that prevents Russel's Paradox (recall example 2.5). The Axiom of Replacement states that if F is a function then for any set X there is a set Y = F(X) = $\{F(x) : x \in X\}$. This is clearer if you consider $\beta(x, y)$ to be associated with a mapping $x \to y$, that is, $\beta_2^{ZF} = \{(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n), \cdots : x_i \neq x_j \textit{ for all } i, j\}$. Thus the antecedent of the axiom implies every element is mapped to exactly one other element and the consequent ensures the existence of a subset for which the mapping is surjective.

Recall that a model in a non-empty set U is a system where every predicate has an n-ary relation assigned to it. By ZF6, it is known ZF contains at least one set X, so it is non-empty. An n-ary relation is a set, thus an assignment to a predicate is merely a set associated with a symbol. For example, ZF gives the symbol $\emptyset$ the assignment $\{y : y \neq y\}$. Thus a model in ZF is simply some system which defines at least one set without contradicting the axioms of ZF. Such definitions are given axiomatically in examples 3.48 through 3.50.

Example 3.48 The Grothendieck Universe [9] is a model in the universe of Zermelo-Fraenkel set theory which gives a set $\mathbb{U}$ with the following properties:

1. $\forall x \forall y (x \in \mathbb{U} \wedge y \in x \Rightarrow y \in \mathbb{U})$ $\quad$ $\mathbb{U}$ *is a transitive set*
2. $\forall x \forall y (x \in \mathbb{U} \wedge y \in \mathbb{U} \Rightarrow \{x, y\} \in \mathbb{U})$ $\quad$ $\mathbb{U}$ *contains all pairs*
3. $\forall x (x \in \mathbb{U} \Rightarrow P(x) \in U)$ $\quad$ $\mathbb{U}$ *contains all power sets*
4. $\forall X \forall u_x (X \in U \wedge \{u_x : x \in X\} \Rightarrow \bigcup_{x \in X} u_x \in \mathbb{U})$
   where $\{u_x : x \in X\}$ *is a collection of elements of* $\mathbb{U}$

Example 3.49 In the universe of Zermelo-Fraenkel Set theory, the von Neumann Universe [15] is a model which gives a hierarchy of sets, denoted $\mathcal{V}$, defined as follows:

- $\mathcal{V}_0$ is the empty set $\emptyset = \{\}$
- For an ordinal number β (well-ordered set β), $\mathcal{V}_{\beta+1} = P(\mathcal{V}_\beta)$
- For any limit ordinal α, $\mathcal{V}_\alpha = \bigcup\limits_{\beta<\alpha} \mathcal{V}_\beta$
- $\mathcal{V} = \bigcup\limits_{\alpha} \mathcal{V}_\alpha$

It is interesting to note that for natural rather than ordinal numbers, $\mathcal{V}_{\mathbb{N}}$ is the set of hereditarily finite sets (hereditarily finite sets are defined recursively where, given that $\emptyset$ is a hereditarily finite set, if $s_1, s_2, \cdots, s_n$ are hereditarily finite sets then $\{s_1, s_2, \cdots, s_n\}$ is also hereditarily finite). However $\mathcal{V}_{\mathbb{N}}$ is a model of ZF only when the Axiom of Infinitiy is negated.

Example 3.50 The Constructible Universe $\mathcal{L}$ [2] is a model of ZF created by Kurt Gödel. It is constructed similarly to the von Nuemann Universe except $\mathcal{V}_{\beta+1}$ is more strictly defined. $\mathcal{L}$ is defined recursively as follows:

- $\mathcal{L}_0$ is the empty set $\emptyset = \{\}$
- For an ordinal number β and a wff γ of first order logic,
  $\mathcal{L}_{\beta+1} = \{y : y \in \mathcal{L}_\beta \wedge (\gamma(y, z_1, z_2, \cdots, z_n) \wedge z_1, z_2, \cdots, z_n \in \mathcal{L}_\beta\}$
- For any limit ordinal α, $\mathcal{V}_\alpha = \bigcup\limits_{\beta<\alpha} \mathcal{V}_\beta$
- $\mathcal{L} = \bigcup\limits_{\alpha} \mathcal{L}_\alpha$

Verbally, the second line says $\mathcal{L}_{\beta+1}$ is constructed from the subsets of the previous stage which can be defined by a wff whose parameters are from, and quantifiers affect, only the previous stage. Gödel proved both the Axiom of Choice and the Generalized Continuum Hypothesis were true in this model.

A group $(G, *)$ is a set G with a binary operation $*$ which satisfies the following four group axioms:

Group Axioms (GA):
GA1: $\forall a \forall b(a, b \in G \Rightarrow a * b \in G)$ *Closure*
GA2: $\forall a \forall b \forall c(a, b, c \in G \Rightarrow (a * b) * c = a * (b * c))$ *Associativity*
GA3: $\forall a(a, e \in G \Rightarrow e * a = a = a * e)$ *Identity*
GA4: $\forall a \exists b(a, b \in G \Rightarrow a * b = e = b * a)$ *Inverse*

Group theory is a universe which contains elements a, b, c, etc, as well as a constant e, all of which comply with the above axioms. The language of GA includes a binary operation $*$ as well as these previously mentioned elements.

Closure ensures that any two elements from G which are combined by the binary operation produces an element which is also in G. Associativity ensures the order in which the binary operation performs is irrelevant as long as the sequence of elements it acts on remains unchanged. Identity ensures the existence of an element which does not affect any other element in the binary operation. Inverse establishes that every element has a counterpart such that their combination under the binary operation produces the identity.

A model in group theory is a specific set which can be acted upon under a specified binary operation, i.e. a group is a model of group theory.

Example 3.51 The integers, $\mathbb{Z}$, under addition, $+$, form a group (G, $+$). This is evident as for all integers a, b, and c

- $a + b \in \mathbb{Z}$
- $(a + b) + c = a + (b + c)$
- $0 + a = a + 0 = a$
- $a + (-a) = (-a) + a = 0$

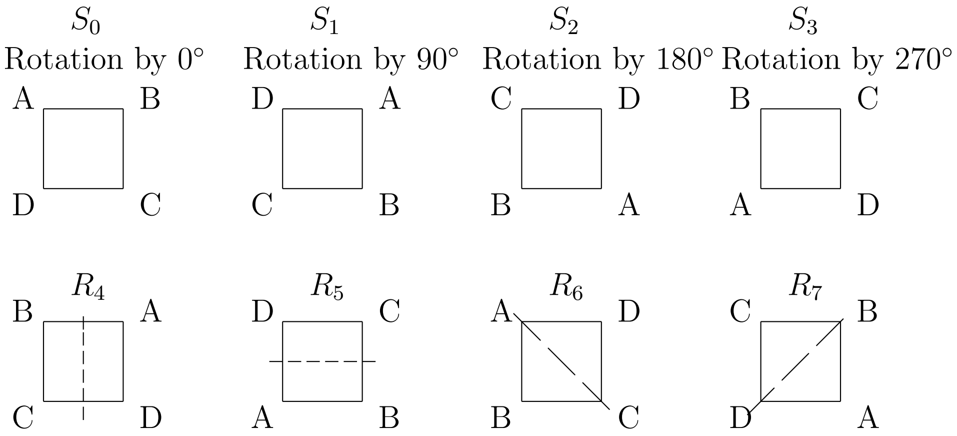
Indeed the rationals, real, and complex numbers also form a group under addition. However, the naturals do not as they lack inverses. Similarly, the integers under multiplication (G, $\cdot$) do not form a model of group theory as not every element has a multiplicative inverse. For example, if $a = 5$, it's inverse would be $\frac{1}{5}$ which is not an integer.

Example 3.52 The integers without 0 modulo p for p a prime form a group ($\mathbb{Z}_p$, $\cdot$) under multiplication. The reasoning follows from the fact $a_1 = a_2$ mod p implies $gcd(a_1, p) = gcd(a_2, p) = 1$. Thus for a, b, and c in $\mathbb{Z}_p$:

- (a mod p) $\cdot$ (b mod p) = ($a \cdot b$) mod p $\in \mathbb{Z}_p$
- ((($a \cdot b$) mod p) $\cdot$ c) mod p = ($a \cdot$ (($b \cdot c$) mod p) mod p
- $a \cdot 1 = 1 \cdot a = a$ (as $\forall p, 1 \in \mathbb{Z}_p$)
- For every a there exists an x such that $a \cdot x = 1$ mod p. Such an x can be found by solving $ax - np = 1$ where $gcd(x, n) = 1$ and n is some natural number.

The reasoning for this last line becomes more evident when a specific model such as $\mathbb{Z}_5 = \{1, 2, 3, 4)$ is considered. The inverse for 1 is, of course 1. For 2 it is necessary to solve the equation $2x - n5 = 1$. If $n = 1$, $2x - 5 = 1$ implies $x = 3$. $3 \in \mathbb{Z}_5$ thus we have the inverse for 2 (The inverse of 3 will then be 2). For 4, the equation becomes $4x - n5 = 1$. However, $x \notin \mathbb{Z}_5$ for $n = 1, 2$, or 3. Only when $n = 4$ does $x = 4 \in \mathbb{Z}_5$. Thus the inverse of 4 is 4.

Example 3.53 . The Diehdral group D_n is a model of group theory which gives the symmetries (rotation and reflection) of a regular polygon. For example, D_4 is the 8 element Diehdral group of a square, containg four rotations R_0, R_1, R_2, and R_3 and four reflections S_4, S_6, S_6, and S_7, illustrated below:



The binary operation for the model is composition. For example, $S_1 \circ S_1 = S_2$, $R_4 \circ R_7 = S_3$, and $S_1 \circ R_6 = R_4$.

Peano's axioms form a universe which includes a constant 0 and an infinite number of elements $s(0)$, $s(s(0))$, $s(s(s(0)))$, etc, where $s(x)$ is the successor function defined by the first order wff $\forall x(x \in S \Rightarrow s(x) \in S)$. The language of PA includes these elements as well as the symbols + and $\cdot$, all of which satisfy the following axioms:

Axioms of Peano Arithmetic (PA):

PA1: $\forall x(\neg s(x) = 0)$ *There is no element in S whose successor is 0*

PA2: $\forall x \forall y(s(x) = s(y) \Rightarrow x = y)$ *s is an injection*

PA3: $\forall y_1 \forall y_2 \cdots \forall y_n(\alpha(0) \wedge \forall x(\alpha(x) \Rightarrow \alpha(s(x))) \Rightarrow \forall x \alpha(x))$
where $\alpha(x)$ is a wff of the Peano Axioms whose free variables
are among $x, y_1, y_2, \cdots, y_n$. *First Order Induction Axiom*

PA4: $\forall x(x + 0 = x)$ *Recursive definitions of +*
PA5: $\forall x \forall y(x + s(y) = s(x + y))$

PA6: $\forall x(x \cdot 0 = 0)$ *Recursive definitions of $\cdot$*
PA7: $\forall x \forall y(x \cdot s(y) = (x \cdot y) + x)$

The first two postulates, in combination with the constant 0 and the definition of the successor function $s(x)$, are known as the Peano Axioms, and explicitely define the universe PA. This is done by first assuming that the set PA is non-empty and drawing the conclusion that it must contain some constant element which can be denoted by the symbol 0. Next, a successor function s is defined such that each element x of PA is associated with a single element of PA given by the successor function, denoted $s(x)$. It is important to note that the successor function is closed in PA as for every element $x \in PA$, $s(x)$ is also in PA. Axiom 1 simply states 0 is not the successor of any element. Axiom 2 states that if two elements have the same successor, then they must be the same.

Although these axioms show PA contains the infinite subset $\{0, s(0), s(s(0)), s(s(s(0))), \cdots\}$ for which no two elements are equal, it does not prove every element of PA is in the set. This is given by PA3, known as the First Order Induction Axiom. This is a weaker version of Peano's original Induction Axiom which states the folllowing:

PA5: If U is a set such that:

- $0 \in U$
- $\forall x(x \in U \Rightarrow s(x) \in U)$

Then U contains every element of S.

However, as U is undefined, in first order logic it is impossible to apply quantifiers. Thus the First Order Induction Axiom defines the set via a predicate α. Recall that $\alpha(x)$ is true if x is an element of some set α^{PA} defined by the unary relation on α. Thus $\alpha(0)$ says $0 \in \alpha^{PA}$ and $\alpha(x) \Rightarrow \alpha(s(x)))$ mean $x \in \alpha^{PA} \Rightarrow s(x) \in \alpha^{PA}$. Thus the First Order Induction Axiom can replace the original to provide a set of axioms expressed in first order logic.

Axioms 4-7 give a recursive definition of $+$ and $\cdot$. Both symbols correspond to a mapping $PA \times PA \to PA$. A model in PA consists of the elements of PA (although they may be represented differently, i.e. $s(0) = 1$), a constant $0 \in PA$ and a successor function s (all of which must satisfy the axioms). There are both standard and non-standard models for Peano Arithmetic.

Example 3.54 The standard model of Peano Arithmetic is the natural numbers ℕ={0, 1, 2, 3, $\cdots$} where addition and subtraction are as defined by basic arithmetic and the successor function is defined such that $s(x) = x + 1$, that is $s(0) = 1$, $s(s(o)) = 2$, $s(s(s(0))) = 3$, etc. This means wffs such as $1 + 1 = 2$ can be proven using the Peano Axioms (as will be shown in section 4.5).

Now the orginal second-order logic Peano Axioms established that, up to isomorphism, the natural numbers where the only model of PA. However, first order logic allows for other models of PA to exists. Any such model contains at least a set which, when linearly ordered (i.e. 0, s(0), s(s(0)), etc) is isomorphic to ℕ. A model which contains additional elements is called non-standard. The Löwenheim-Skolem theorem and the compactness theorem of model theory both guarantee the existence of non-standard models.

Now that the axiomatic systems of Zermelo-Fraenkel set thoery, group theory, and Peano Arithmetic have been outlined and explained, along with various models, the next step is to provide a proof for the theorems (valid wffs) of each system. The semantic tableau gives such a proof.

Chapter 4

The Semantic Tableau

This chapter will use the commonalities of sentential and first order logic to give definitions, examples, and proofs which will apply to both logics simultaneously. These shared concepts will be used to explore the notion that a wff can be a semantic consequence of a set of wffs, i.e. true in the same models. The idea of contradictory and finished sets will also be introduced, along with a theorem that states every finished set has a model. The information of these first three sections will lay the groundwork for the construction of semantic tableaus in the final section. This last section will also show how semantic tableaus can prove a wff is a semantic consequence as well as a wff's validity.

4.1 Reconciling Sentential and First Order Logic

Sentential and first order logic share several aspects in common, including propositions, connectives, and truth values from the set $\{\mathbf{T}, \mathbf{F}\}$. This section will show how these shared aspects can be used to express definitions, examples, and even proofs simultaneously for both logics.

At the end of section 3.2 the 0-ary predicate P_0 and its 0-ary relation

P_0^U was briefly discussed. The predicate P_0 is the proposition of first order logic; an expression with no variables that has a truth value in a model defined by the relation P_0^U. While a sentential logic model sends a proposition P to the set $\{\mathbf{T},\ \mathbf{F}\}$, a first order logic model in a universe U sends P_0^U to the set $\{\{()\},\ \emptyset\}$. As any n-ary relation $P_n^U = \emptyset$ gives an evaluated predicate $P_n(a_1, a_2, \cdots, a_n)$ a truth value of $\mathbf{F}$ for $(a_1, a_2, \cdots, a_n) \in U^n$, $P_0^U = \emptyset$ means $P_0()$ is false. On the other hand, the relation $P_0^U = \{()\}$ indicates $P_0()$ is true as $() \in \{()\}$. Thus there is a mapping $P_0^U \to \{\{()\}, \emptyset\} \to \{\mathbf{T}, \mathbf{F}\}$ that resembles the sentential mapping $P \to \{\mathbf{T}, \mathbf{F}\}$.

This implies sentential logic can be described in a similar way as first order logic, i.e. many definitions of first order logic can be used simlutaneously for both logics. For example, a sentential wff is valid by definition 3.45 if it is a tautology. Similarly, definition 3.39 of a model can be used universally, where, in sentential logic, every ‘predicate’ has 0-arity, the only relation is the 0-ary relation given above, and U is inconsequential (as only $U^0 = \{\{()\}$ is considered). Though it is important to realize there are two ways “model” can be used in association with a wff or set of wffs. A wff α can be considered in a particular model M and may be either true or false in that model. M may also model α, or be a model of α, both of which imply α is true in M and are denoted by the turnstile symbol $M \models \alpha$.

As a proposition P is a predicate P_0, the recursive definitions 2.25 and 3.24 which create sentential and first order logic wffs imply every wff of sentential logic is also a wff of first order logic (although the reverse is certainly not true). Thus wffs which are valid in sentential logic are also valid in first order logic. For the remainder of this paper, if a definition, example, or theorem uses wffs that are not specified as being of a particular logic, it is considered to apply in both logics (i.e., if an example uses two wffs α and β, they can both be wffs of sentential logic or both wffs of first order logic). This allows many proofs to be condensed, as much of what composes a proof in sentential logic is needed (but not suffices) to prove its counterpart in first order logic.

This compression of proofs will be convenient as ideas such as semantic consequence, finished sets, and tree diagrams must be explored before semantic tableaus can even be considered.

4.2 Semantic Consequence

Definition 4.1 A wff α is a **semantic consequence** of a set of wffs H, denoted $H \models \alpha$, if every model of H is a model of α.

Deductive reasoning is based on the idea a conclusion is a semantic consequence of some premises. Modus ponens uses that a wff β is a semantic consequence of wffs α and $(\alpha \Rightarrow \beta.)$ This follows, because if α and $(\alpha \Rightarrow \beta)$ are true in a model M, by propositions 2.48 and 3.44, so is β. Similarly, in modus tollens, the wff $\neg\alpha$ is a semantic consequence of wffs $\neg\beta$ and $(\alpha \Rightarrow \beta)$.

Example 4.2 Let β_1 and β_2 be true wffs in a model M. Then $\alpha = (\beta_1 \Leftrightarrow \beta_2)$ is also a true wff in M and thus is a semantic consequence of β_1 and β_2 by propositions 2.48 and 3.44. □

Example 4.3 Consider the sentential logic wffs $(p \vee q)$ and $(r \Rightarrow s)$. By the truth table given in example 2.49, it is evident the wff $((p \vee q) \wedge (r \Rightarrow s))$ is a semantic consequence of those wffs, as every model in which they are true, $((p \vee q) \wedge (r \Rightarrow s))$ is also true.

Example 4.4 Consider the sentential logic wff $\alpha : ((p \wedge q) \wedge r)$, where p, q, and r are propositions and the set $H = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ where

$\beta_1 : (p \Rightarrow q)$
$\beta_2 : (q \Rightarrow r)$
$\beta_3 : (p \Leftrightarrow r)$
$\beta_4 : (p \vee q)$

Does $H \models \alpha$? It is necessary to conside all possible models for p, q, and r, as illustrated in the table, to show any model M_i in which every wff of H is satisfied, α is as well. As β_1 is assumed to be true, this wff implies $p_{M_i} = \mathbf{T}$ and $q_{M_i} = \mathbf{F}$ is impossible for the same model i. Thus M_2 and M_6 are eliminated. Similary, β_2 disqualifies M_5 and M_6. β_3 indicates $p_{M_i} = r_{M_i}$ so M_3, M_4, M_5 and M_6 are eliminated. β_4 indicates both q_{M_i} and r_{M_i} cannot be false, eliminating M_6 and M_8.

	p	q	r	Disqualification
M_1	T	T	T	
M_2	T	F	T	β_1
M_3	F	T	T	β_3
M_4	F	F	T	β_3, β_4
M_5	T	T	F	β_3, β_2
M_6	T	F	F	β_1, β_3, β_4
M_7	F	T	F	β_2
M_8	F	F	F	β_4

Thus M_1 is the only model in which every wff of H is true (Note that every wff β_i was required to eliminate the other models). Thus $p_{M_1} = q_{M_1} =$

$r_{M_1} = \mathbf{T}$ and the wff $\alpha : ((p \wedge q) \wedge r)$ is true when β_1, β_2, β_3, and β_4 are. Thus α is a semantic consequence of the wffs β_1, β_2, β_3, and β_4 and $H \models \alpha$. □

The turnstile notation $\models$ can be used in three ways, so it is important to realize what the notation is symbolizing. If M is a model, H a set of wffs, and α a single wff, then:

- $M \models \alpha$ indicates M is a model of α, i.e. α is true in M.
- $M \models H$ indicates M is a model of every wff in H.
- $H \models \alpha$ indicates every model of H is a model of α.

Validity can be stated using semantic consequence. Now a wff α is valid if it is true in every model, while α is a semantic consequence of set off wffs H if it is true in the same models as H. The empty set $\emptyset$ is true in every model, thus if $\emptyset \models \alpha$, α is valid.

Suppose a set of wffs H contains both a wff α and its negation $\neg\alpha$. Now, no wff can be a semantic consequence of such as a set, because no model M models both a wff and its negation. This type of set is considered contradictory and is addressed in the next section.

4.3 Finished Sets

This section focuses on the notion of contradictory and finished sets, both of which are necessary in the construction of semantic tableaus. This section will also present a proof that every finished set of wffs has a model.

Definition 4.5 A set W of wffs is **contradictory** if there exists some wff α such that both α and $\neg\alpha$ are in W.

Example 4.6 The set of wffs $\{\neg\alpha, \alpha, \beta, \gamma\}$ is contradictory while the sets of wffs $\{\alpha, \beta, \gamma\}$ and $\{\neg\alpha\,\beta, \gamma\}$ are not. □

Definition 4.7 A **basic** wff is a simple wff or the negation of a simple wff.

A basic wff of sentential logic is then either a proposition or the negation of a proposition. Similarly, a basic wff of first order logic is a n-ary predicate (n potentially 0) or the negation of an n-ary predicate.

Definition 4.8 A set W of wffs is **finished** if it is not contradictory and for each wff γ in W, either γ is basic or one of the following is true:

$\neg\neg$: γ has the form $\neg\neg\alpha$, where $\alpha \in W$.

$\wedge$: γ has the form $(\alpha \wedge \beta)$, where $\alpha \in W$ and $\beta \in W$.

$\neg\wedge$: γ has the form $\neg(\alpha \wedge \beta)$ where either $\neg\alpha \in W$ or $\neg\beta \in W$.

$\vee$: γ has the form $(\alpha \vee \beta)$, where either $\alpha \in W$ or $\beta \in W$.

$\neg\vee$: γhas the form $\neg(\alpha \vee \beta)$, where both $\neg\alpha \in W$ and $\neg\beta \in W$.

$\Rightarrow$: γ has the form $(\alpha \Rightarrow \beta)$, where either $\neg\alpha \in W$ or $\beta \in W$.

$\neg \Rightarrow$: γ has the form $\neg(\alpha \Rightarrow \beta)$, where both $\alpha \in W$ and $\neg\beta \in W$.

$\Leftrightarrow$: γ has the form $(\alpha \Leftrightarrow \beta)$, where either both $\alpha \in W$ and $\beta \in W$, or both $\neg\alpha \in W$ and $\neg\beta \in W$.

$\neg \Leftrightarrow$: γ has the form $\neg(\alpha \Leftrightarrow \beta)$, where either both $\alpha \in W$ and $\neg\beta \in W$, or both $\neg\alpha \in W$ and $\beta \in W$.

If W is a set of wffs of first order logic in a universe U, every wff $\gamma \in W$ must have no parameters or free variables and, if γ is not basic, one of the following may be true:

$\forall$: γ has the form $\forall x\alpha$ where $\alpha(x//a) \in W$ for every $a \in U$.

$\neg\forall$: γ has the form $\neg\forall x\alpha$ where $\neg\alpha(x//a) \in W$ for some $a \in U$.

$\exists$: γ has the form $\exists x\alpha$ where $\alpha(x//a) \in W$ for some $a \in U$.

$\neg\exists$: γ has the form $\neg\exists x\alpha$ where $\neg\alpha(x//a) \in W$ for every $a \in U$.

Example 4.9 Consider the set of wffs $\{\alpha, \beta, \neg\gamma, \neg(\alpha \Rightarrow \gamma), (\alpha \vee \neg(\beta \wedge \gamma))\}$. By definition 4.8, this is a finished set as:

- α, β, and $\neg\gamma$ are all basic wffs.
- For $\neg(\alpha \Rightarrow \gamma)$, both α and $\neg\gamma$ are in the set.
- For $(\alpha \vee \neg(\beta \wedge \gamma))$, α is in the set.

Note that if any one of the basic wffs were missing, this would not be a finished set. □

Example 4.10 Consider the set of wffs $\{\forall x\alpha, \alpha(1), \alpha(2)\}$ in the universe $U = \{1, 2, 3\}$. By definition 4.8, this is not a finished set as $\alpha(3)$ is missing. □

Example 4.11 Consider the set of wffs $\{(\alpha \vee \beta), (\alpha \wedge \beta), \alpha, \neg\alpha, \beta, \}$. By definition 4.8, this is not a finished set as it contains both a wff and its negation.

□

Theorem 4.12 The Finished Set Thoerem: *If W is a finished set of wffs then W has a model.*

Proof: If W is a finished set of wffs (with a universe U in first order logic) then the set containing the basic wffs of W has at least one model M where $\gamma_M = \mathbf{T}$ if $\gamma \in W$ and $\gamma_M = \mathbf{F}$ if $\gamma \notin W$, where γ is a basic wff (γ_M=$\mathbf{T}$ means $\gamma(a_1, a_2, \cdots, a_n) \in \gamma_n^U$ for an n-ary relation specified by M in first order logic, where each $a_i \in U$). This implies γ_M=$\mathbf{F}$ if $\neg\gamma \in W$. Through induction, it can be proven M is a model of the entire set W.

If γ is a basic wff of W, then M models γ by the definition of M given above. Now assume M models every wff in W with length less than γ. Then, as $\gamma \in W$, and W is a finished set, γ must be in one of the thirteen forms of definition 4.8 (the first nine for a sentential finished set) and the following must be true by propositions 2.48 and 3.44:

$\gamma = \neg\neg\alpha$: Then $\alpha \in W$ and, as $M \models \alpha$, $M \models \neg\neg\alpha$.

$\gamma = (\alpha \wedge \beta)$: Then $\alpha \in W$ and $\beta \in W$, and, as $M \models \alpha$ and $M \models \beta$, $M \models (\alpha \wedge \beta)$

$\gamma = \neg(\alpha \wedge \beta)$: Then either $\neg\alpha \in W$ or $\neg\beta \in W$, and, as $M \models \neg\alpha$ or $M \models \neg\beta$, $M \models \neg(\alpha \wedge \beta)$.

$\gamma = (\alpha \vee \beta)$: Then eitherr $\alpha \in W$ or $\beta \in W$, and, as $M \models \alpha$ or $M \models \beta$, $M \models (\alpha \vee \beta)$.

$\gamma = \neg(\alpha \vee \beta)$: Then both $\neg\alpha \in W$ and $\neg\beta \in W$ and, as $M \models \neg\alpha$ and $M \models \neg\beta$, $M \models \neg(\alpha \vee \beta)$.

$\gamma = (\alpha \Rightarrow \beta)$: Then either $\neg\alpha \in W$ or $\beta \in W$ and, as $M \models \neg\alpha$ or $M \models \beta$, $M \models (\alpha \Rightarrow \beta)$.

$\gamma = \neg(\alpha \Rightarrow \beta)$: Then both $\alpha \in W$ and $\neg\beta \in W$, and, as $M \models \alpha$ and $M \models \neg\beta$, $M \models \neg(\alpha \Rightarrow \beta)$.

$\gamma = (\alpha \Leftrightarrow \beta)$: Then either both $\alpha \in W$ and $\beta \in W$, or both $\neg\alpha \in W$ and $\neg\beta \in W$, and, as $M \models \alpha$ and $M \models \beta$ or $M \models \neg\alpha$ and $M \models \neg\beta$, $M \models (\alpha \Leftrightarrow \beta)$.

$\gamma = \neg(\alpha \Leftrightarrow \beta)$: Then either both $\alpha \in W$ and $\neg\beta \in W$, or both $\neg\alpha \in W$ and $\beta \in W$ and, as $M \models \alpha$ and $M \models \neg\beta$ or $M \models \neg\alpha$ and $M \models \beta$, $M \models \neg(\alpha \Leftrightarrow \beta)$.

$\gamma = \forall x\alpha$: Then $\alpha(x//a) \in W$ for every $a \in U$ and, as $M \models \alpha(x//a)$ for every a, $M \models \forall x\alpha$

$\gamma = \neg\forall x\alpha$: Then $\neg\alpha(x//a) \in W$ for some $a \in U$ and, as $M \models \neg\alpha(x//a)$ for some a, $M \models \neg\forall x\alpha$.

$\gamma = \exists x\alpha$: Then $\alpha(x//a) \in W$ for some $a \in U$, and, as $M \models \alpha(x//a)$ for some a, $M \models \exists x\alpha$.

$\gamma = \neg\exists x\alpha$: Then $\neg\alpha(x//a) \in W$ for every $a \in U$ and, as $M \models \neg\alpha(x//a)$ for every a, $M \models \neg\exists x\alpha$.

Thus, by induction, M models every wff in W, so M models W (Note that the model of this proof is constructed based on the basic wffs of W, so the proof also shows that any model of those basic wffs in W is a model of all the wffs in W). ∎

Now that the notions of contradictory and finished sets have been defined, it is possible to use them to develop semantic tableaus.

4.4 Semantic Tableaus

Semantic Tableaus, also known as truth trees, are decision making and proof procedures for sentential and first order logic. They are often easier to evaluate than a truth table and are not limited to sentential logic as the tables are. Tableaus can build a model of a set of wffs, show how one wff is a semantic consequence of others, and prove a wff is valid (although all but the last will be explored in the next chapter). The term "semantic" is used as it is indicative of the fact wffs can take different truth values in different models "Semantic tableau" is often shortened to simply "tableau," where the term is implied rather than stated. This section will develop the notion of tree diagrams and then use them to segue into tableaus.

A wff can be proven valid by either showing it is true in every model or by showing that there is no model in which it is false; in other words, presenting a confutation of the negation of the wff. This is done by assuming the wff is not true and deriving a contradiction. The argument of modus ponens, where $((\alpha \Rightarrow \beta) \land \alpha) \Rightarrow \beta$, can be proven by this method in the following way:

1: First, assume the negation, $\neg(((\alpha \Rightarrow \beta) \land \alpha) \Rightarrow \beta)$ is true in some model. This implies $((\alpha \Rightarrow \beta) \land \alpha) \Rightarrow \beta)$ is false. But this can only happen if the antecedent is true and the consequent is false, i.e.

2: $((\alpha \Rightarrow \beta) \wedge \alpha)$ is true and

3: $\neg\beta$ is true (as β is false). However, if **2** is true, then the wffs joined by the conjunction must both be true, i.e.

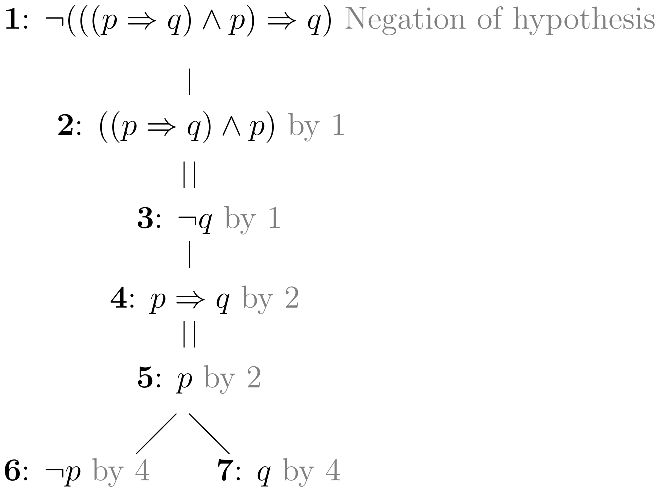
4: $\alpha \Rightarrow \beta$ is true and

5: α is true. However **4** implies either the antecedent is false or the consequent is true, or both. Thus

6: $\neg\alpha$ is true if the antecedent is false, which contradicts **5** and/or

7: β is true if the consequent is true, which contradicts **3**.

Thus there is no model in which modus ponens is not true so it must be true for all models. This reasoning process can be arranged into the following diagram:



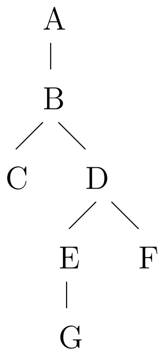
This type of digram is referred to as a tree and can also be used to prove a wff is a semantic consequence of other wffs. Each step of the reasoning process corresponds to a *node* of the tree, the first of which is called the *root*, though it is traditionally drawn at the top. The *parent* of a node t is the node given by the parent function $\pi(t)$. The single and double lines connecting these nodes will be explained later in the section.

Definition 4.13 A **tree** T is a system with a finite or countable number of nodes, a root node r_T, and a parent function π_T, where π_T satisfies
$\pi^0(t) = t$, $\pi^1(t) = \pi(t)$, $\pi^2(t) = \pi(\pi(t))$, $\pi^3(t) = \pi(\pi(\pi(t)))$, $\cdots$, $\pi^n(t) = r_T$.

This last part simply says that eventually the parent function, when applied to a node t, and then to t's parent, and then to t's parent's parents, etc, will eventually reach the root node. It is helpful to distinguish between the various types of nodes and the qualities they possess. A root and parent node have already been described, but there are also ancestor, child, and bachelor nodes.

The **ancestors** of a node t are the nodes $\pi^0(t)$, $\pi^1(t)$, $\pi^2(t)$, etc, while the **proper ancestors** of t are the nodes $\pi^1(t)$, $\pi^2(t)$, $\pi^3(t)$, etc.Thus the root is an ancestor of every node, including itself, and a proper ancesotor for every node but itself. A **child node** t is a node with a parent $\pi(t)$ while a **grandchild node** t has a grandparent $\pi(\pi(t))$. A **bachelor node**, or terminal node, is a node with no children. Often nodes appear in patterns that resemble branches on a tree, and can be grouped by this characteristic. Γ is a **branch** of a tree T if Γ is a subset of T which includes the root node and the parent of each nonroot node already in Γ, and each node in Γ is either a bachelor or has exactly one child in Γ. Thus a branch of a tree will either have a single bachelor node, and be finite, or no bachelor nodes at all, in which case it will be infinite.

Example 4.14 Consider how the previous definitions apply in the following tree.



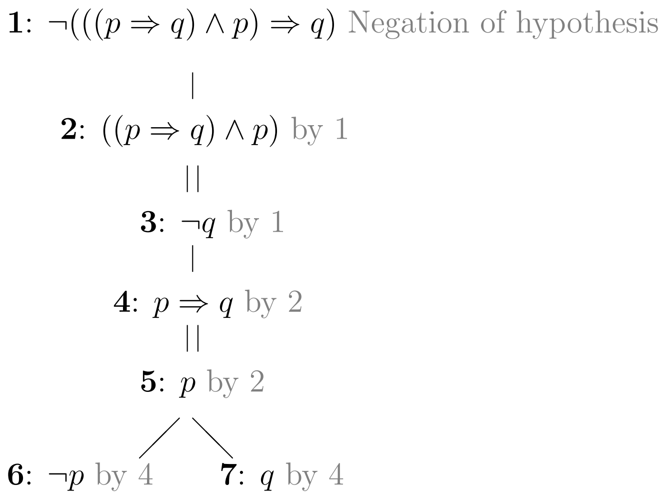
A is the root of the tree, whose parent function is defined by π(G) = E, π(E) = π(F) = D, π(C) = π(D) = B, and π(B) = A. This means a node such as C has the ancestors C, B, and A and the proper ancestors of the node G are E, D, B, and A. The node D is the child of B and the grandchild of A, while C, F, and G are all bachelor nodes. The branches of this tree are (A, B, C), (A, B, D, E, G), and (A, B, D, F). □

Definition 4.15 A **labeled tree** is a system with a tree T, a finite or countable set of wffs H called the hypotheses, and a wff $\phi(t)$ corresponding to each nonroot node t.

When $\alpha = \phi(t)$, the wff α is said to occur at t, or is t. The wffs of H are considered to occur at the root node.

Definition 4.16 Γ is a **finished branch** of a labeled tree T if the set of wffs corresponding to the nodes of the branch is either finished. Γ is a **contradictory branch** if that set of wffs is contradictory.

Example 4.17 Consider the tree diagram at the beginning of this chapter, given again here.



This is a labeled tree whose root was the hypothesis set that contained only the negation of modus ponens. Its two branches (1, 2, 3, 4, 5, 6) and (1, 2, 3, 4, 5, 7) were contradictory. Its parent function was defined by $\pi(6) = \pi(7) = 5$, $\pi(5) = 4$, $\pi(4) = 3$, $\pi(3) = 2$, and $\pi(2) = 1$. The ancestors of 6 included every wff but 7, while the ancestors of 7 included every wff but 6. Both 6 and 7 were bachelor nodes while every other node was a parent (Note that the numbers actually represent their corresponding wffs on the labeled tree). □

A labeled tree can be of sentential logic or first order logic, depending on the type of wffs used. Similarly, a tableau chain or tableau can be solely of sentential logic, or expand to cover first order logic.

Definition 4.18 A **sentential logic tableau chain** is a finite or infinite sequence of finite labeled trees T_0, T_1, T_2, $\cdots$, T_n, $\cdots$ where

- T_0 is a single root node with the hypotheses set H,
- each T_{k+1} is found by applying a tableau extension rule at a bachelor node t of Γ_k of T_k, where Γ_k is not contradictory,
- T_n is the last labeled tree of a finite tableau chain only if every branch of T_n is finished or contradictory, and,
- for α and β wffs of sentential logic, the following are the tableau extension rules:

$\neg\neg$: If t has an ancestor $\neg\neg\alpha$, add a child α of t.

$\wedge$: If t has an ancestor $\alpha \wedge \beta$, add a child α and a grandchild β of t.

$\neg\wedge$: If t has an ancestor $\neg(\alpha \wedge \beta)$, add two children $\neg\alpha$ and $\neg\beta$ of t.

$\vee$: If t has an ancestor $\alpha \vee \beta$, add two children α and β of t.

$\neg\vee$: If t has an ancestor $\neg(\alpha \vee \beta)$, add a child $\neg\alpha$ and a grandchild $\neg\beta$ of t.

$\Rightarrow$: If t has an ancestor $\alpha \Rightarrow \beta$, add two children $\neg\alpha$ and β of t.

$\neg \Rightarrow$: If t has an ancestor $\neg(\alpha \Rightarrow \beta)$, add a child α and a grandchild $\neg\beta$ of t.

$\Leftrightarrow$: If t has an ancestor $\alpha \Leftrightarrow \beta$, add two children α and $\neg\alpha$ of t, a child β of α, and a child $\neg\beta$ of $\neg\alpha$.

$\neg \Leftrightarrow$: If t has an ancestor $\neg(\alpha \Leftrightarrow \beta)$, add two children α and $\neg\alpha$ of t, a child $\neg\beta$ of α, and a child β of $\neg\alpha$.

Definition 4.19 For a **first order logic tableau chain** it is neccessary to add the following to the above definition, where

- α and β are now wffs of first order logic,
- x is a variable,
- a and b are either variables or parameters,
- b does not occur in any ancestor of the bachelor node t, and
- it is necessary to add the following tableau extension rules:

$\forall$: If t has an ancestor $\forall x\alpha$, add a child $\alpha(x//a)$ of t.

$\neg\forall$: If t has an ancestor $\neg\forall x\alpha$, add a child $\neg\alpha(x//b)$.

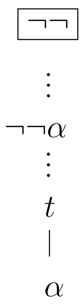
$\exists$: If t has an ancestor $\exists x\alpha$, add a child $\alpha(x//b)$.

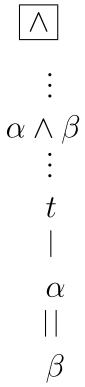
$\neg\exists$: If t has an ancestor $\neg\exists x\alpha$, add a child $\neg\alpha(x//a)$.

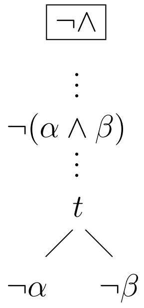
(Note that there may be infinitely many choices for a and b).

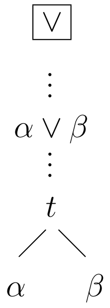
Pictorially, the tableau extension rules can be shown as follows, where if $\boxed{}$ is an ancestor, then a terminal node t can be extended to have a child, marked by a single line, or a grandchild, marked by a double line.

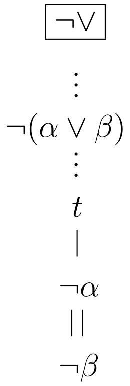
Tableau Extension Rules

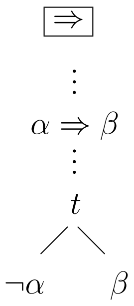


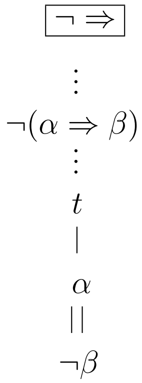


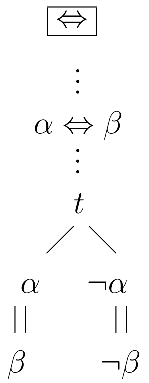


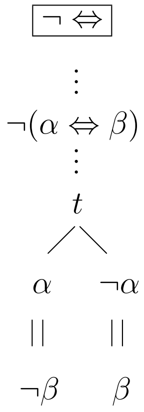


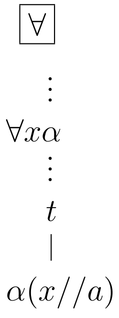


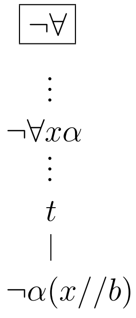


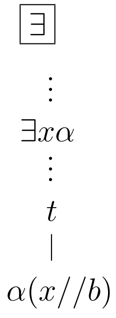


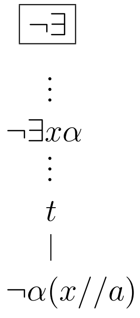




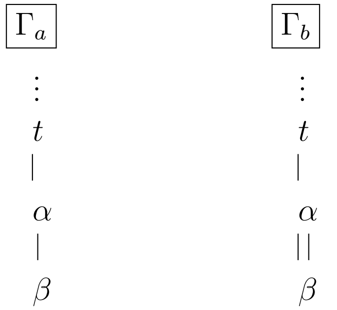








The single and double line notation helps identify which tableau extensions are used. For example, consider Γ_a and Γ_b, two branches of a tableau which have been exteded from some node t as shown below:



The single lines used in Γ_a indicate two tableau extensions $\boxed{\neg\neg}$ were used on some ancestors $\neg\neg\alpha$ and $\neg\neg\beta$, while the single and double line combo of Γ_b indicate a single tableau extensions $\boxed{\wedge}$ was used on an ancestor $\alpha \wedge \beta$.

Definition 4.20 A wff α corresponding to an ancestor node is **used** if a tableau extension rule has been applied to it in order to extend through a bachelor node t. α is **unused** if it is not basic and there is a noncontradictory branch through t on which α is not used.

Thus T_n is the last labeled tree of a finite tableau chain only there are no unused wffs (However, note that a 'used' wff which has extended a bachelor node may still be 'unused' in the tableau as a whole).

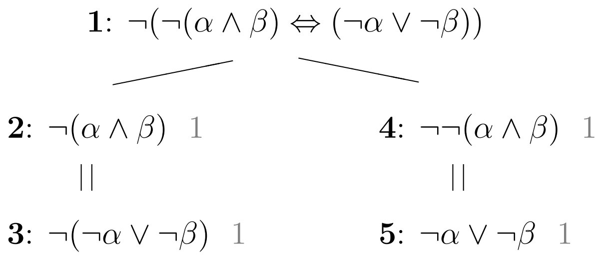
Example 4.21 Consider the tableau chain of the negation of DeMorgan's law second law, $\neg(\neg(\alpha\wedge\beta) \Leftrightarrow (\neg\alpha\vee\neg\beta))$, where α and β are wffs. The first labeled tree, T_0, is the hypothesis set. In this case, it is simply the negation of the law.

$$\boxed{\mathrm{T}_0}$$

$$\mathbf{1}\text{: } \neg(\neg(\alpha \wedge \beta) \Leftrightarrow (\neg\alpha \vee \neg\beta))$$

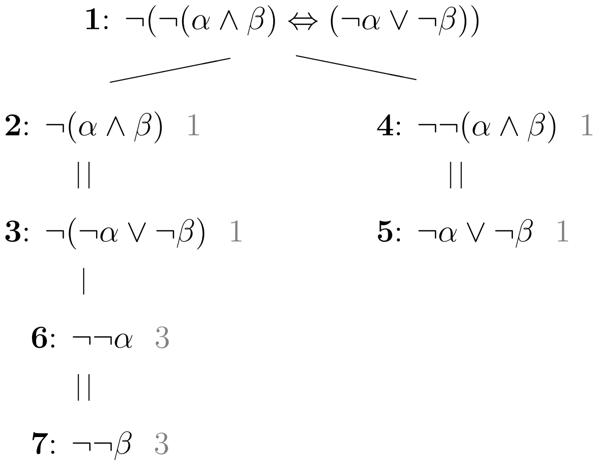
T_1 is found by applying an tableau extension rule to the terminal node of T_0 which is of the form $\boxed{\neg \Leftrightarrow}$. Thus it is necessary to add two children (**2** and **4**) and two grandchildren (**3** and **5**)

$$\boxed{\mathrm{T}_1}$$



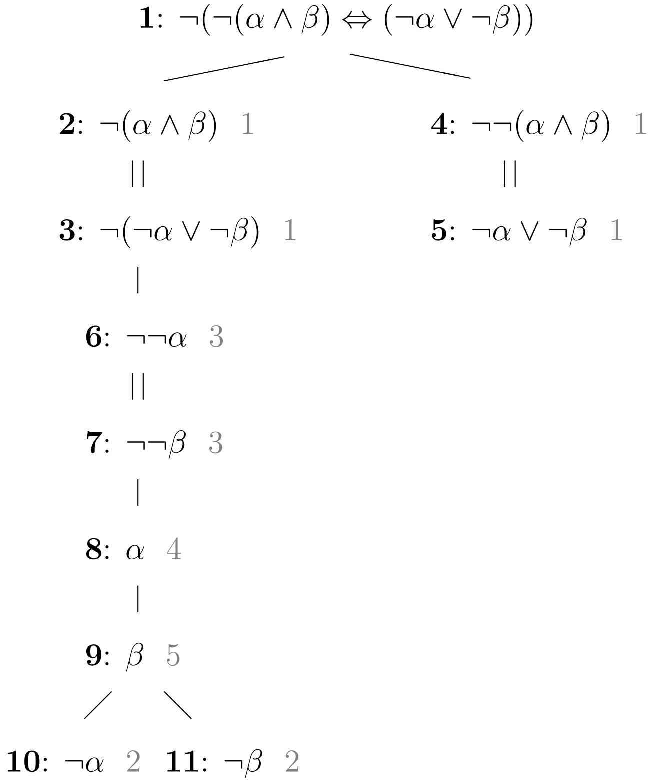
The grayscale numbering next to each node indicates the node it was extended from. There are three ancestor nodes that can be extended at each of the two terminal nodes to form T_2. Extending with an ancestor that has already been extended is allowable, but rather pointless as it simply creates a loop. For simpliciy's sake, the ancestors which do not give a branch will be used first, to avoid creating large branches. Thus T_2 will be extended from the ancestor **3** and terminal node **3** using $\boxed{\neg\vee}$ by adding a child and a grandchild.

$\boxed{\mathrm{T}_2}$

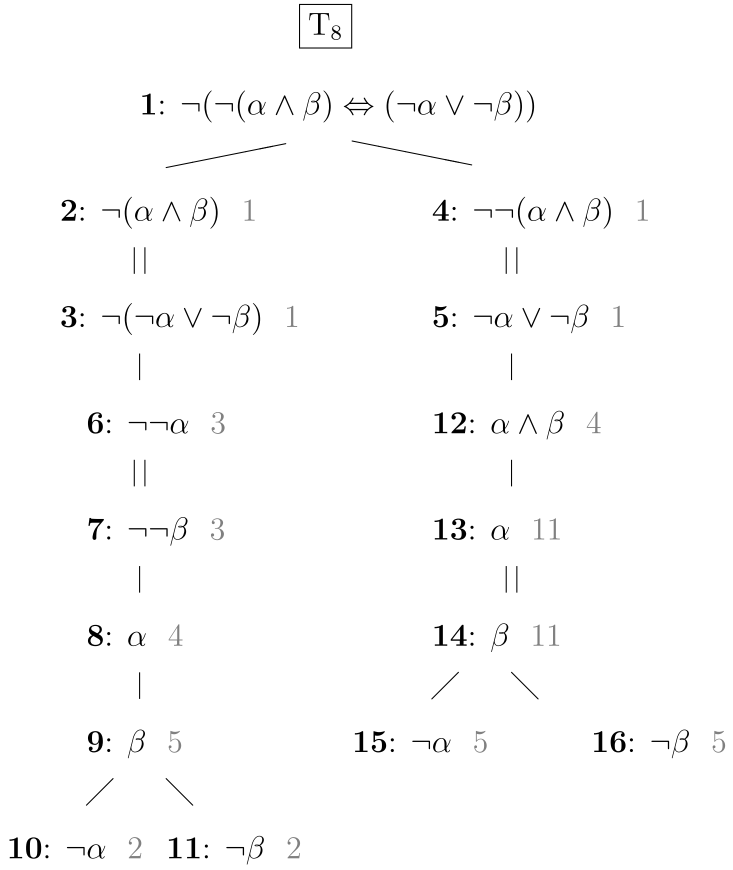


The extension can continue at every terminal node on the left branch until T_5 at which point every node will be used.

$\boxed{\mathrm{T}_5}$



Simlarly, the right branch can be extended in the same way until T_8, where every branch is contradictory.



This sequence of trees, T_0 through T_8 is a tableau chain which applies in both sentential and first order logic. □

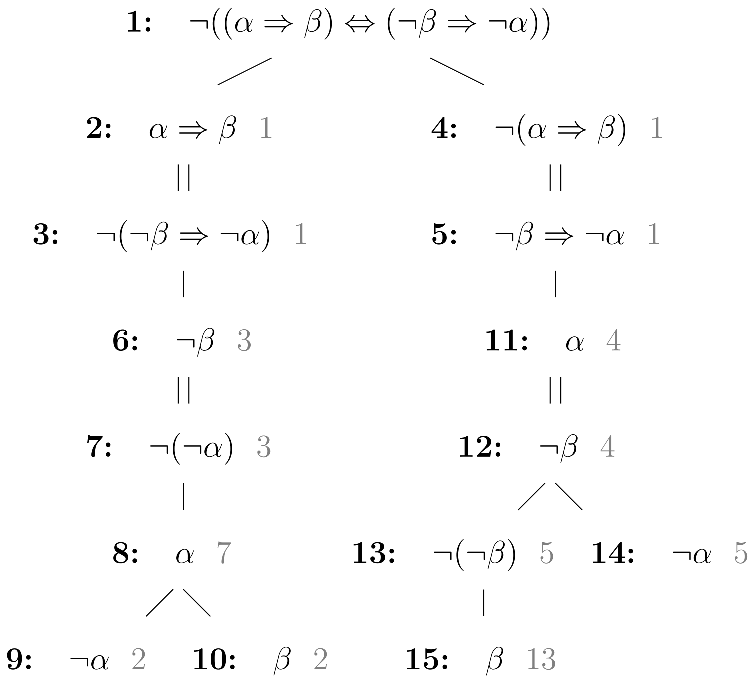
Definition 4.22 A **finite tableau** is a labeled tree T_n which is the last of a finite tableau chain $T_0, T_1, \cdots, T_n$. An **infinite tableau** is a tree which is the union of an infinite tableau chain $T_0, T_1, \cdots, T_n, \cdots$.

T_8 of example 4.21 was a finite tableau. Note that every branch of a finite tableau is either contradictory or finished. A tableau then refers to either a finite or infinite tableau with a hypothesis set H for its root. Although a tableau may be finite, its hypothesis set H may be either finite or countably infinite.

Definition 4.23 A **tableau confutation** of a hypothesis set H of wffs is a finite tableau T with root H where every branch of T is contradictory. A **tableau proof** of a wff α from a hypothesis set H is a tableau confutation of $H \cup \{\neg\alpha\}$.

T_8 of example 4.21 gives a tableau confutation of the one element set $\{\neg(\neg(\alpha \wedge \beta) \Leftrightarrow (\neg\alpha \vee \neg\beta))\}$ as every branch contains both a wff and its negation. When a tableau proof of a wff α from H exist, it is said α is provable from H, denoted $H \vdash \alpha$. Similarly, the notation $\vdash \alpha$ says there exists a tableau proof of α (here $H = \emptyset$). Such a wff is valid (this is proven in the next chapter by the soundness theorem). T_8 of example 4.21 proved the validity of DeMorgan's law. Similarly, the following example proves the validity of the equivalence between an implication and its contrapositive (proven earlier in example 2.54 for sentential logic only).

Example 4.24 Tableau proof that an implication and its contrapositive is equivalent, i.e. $(\alpha \Rightarrow \beta) \Leftrightarrow (\neg\beta \Rightarrow \neg\alpha)$ is valid (in both sentential and first order logic).



Example 4.25 Let $P(x)$ and $Q(x)$ be predicates. Then $\exists x(P(x) \lor Q(x) \vdash (\exists x P(x) \lor \exists x Q(x))$ in first order logic by the following tableau proof.

$\mathbf{1}$: $\neg(\exists x P(x) \lor \exists x Q(x))$ Negation of wff; to be proven

$\mathbf{2}$: $\exists x(P(x) \lor Q(x))$ Hypothesis

$\mathbf{3}$: $\neg\exists x P(x)$ 1

$\mathbf{4}$: $\neg\exists x Q(x)$ 1

$\mathbf{5}$: $\neg P(a)$ 3

$\mathbf{6}$: $\neg Q(a)$ 4

$\mathbf{7}$: $P(a) \lor Q(a)$ 2

$\mathbf{8}$: $P(a)$ 7 $\mathbf{9}$: $Q(a)$ 7

□

Example 4.26 Consider a wff $(\alpha \implies \gamma)$ and the hypothesis set $\{(\alpha \implies \beta), (\beta \implies \gamma)\}$. A tableau proof of the wff from the hypothesis set proves the law of syllogism holds in both sentential and first order logic.

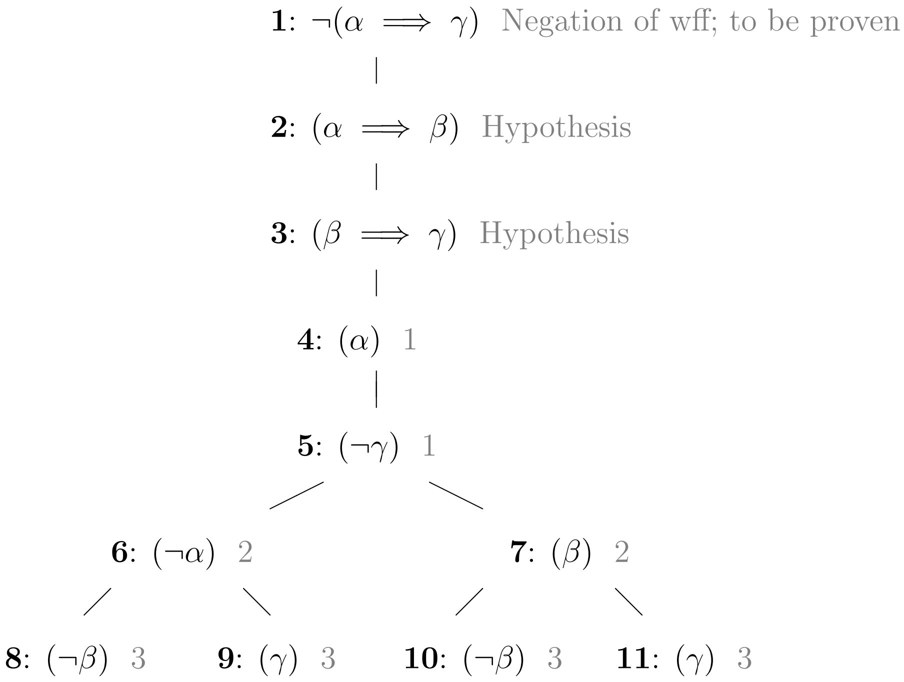
Syllogism, from the greek syllogismos meaning conclusion, inference, or even deduction, is a method of deductive inference introduced by Aristotle in his *Prior Analytics* in the 4th century, B.C. The law of syllogism, also known as hypothetical syllogism, combines two statments involving an implication and a common parameter to draw a third where the antecedent of the first implies the consequent of the second. When the first two statements are true, the third is as well. For example:

1: *If Jenny doesn't work, then she will have no money.* $(\alpha \implies \beta)$
2: *If Jenny has no money, then she can't pay the rent.* $(\beta \implies \gamma)$
3: *If Jenny doesn't work, then she can't pay the rent.* $(\alpha \implies \gamma)$

In sentential and first order logic, the law of syllogism can be expressed as either of the logically equivalent wffs given below:

$$(((\alpha \implies \beta) \land (\beta \implies \gamma)) \Rightarrow (\alpha \implies \gamma))$$
$$((\alpha \Rightarrow \beta) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \gamma)))$$

The following tableau shows the conclusion given by the law of syllogism is provable from its premises, where $\{(\alpha \implies \beta), (\beta \implies \gamma)\} \vdash (\alpha \implies \gamma)$.



Every branch is contradictory, thus $(\alpha \implies \gamma)$ is provable from $\{(\alpha \implies \beta), (\beta \implies \gamma)\}$.

The law of syllogism is also valid, shown by the following tableau proof.

1: $\neg(((\alpha \implies \beta) \wedge (\beta \implies \gamma)) \Rightarrow (\alpha \implies \gamma))$ Negation of wff; to be proven

|

2: $((\alpha \implies \beta) \wedge (\beta \implies \gamma))$ 1

||

3: $\neg(\alpha \implies \gamma)$ 1

|

4: $(\alpha \implies \beta)$ 2

||

5: $(\beta \implies \gamma)$ 2

|

6: α 3

||

7: $(\neg\gamma)$ 3

/ \

8: $\neg\alpha$ 4 **9**: β 4

/ \

8: $\neg\beta$ 5 **9**: γ 5

Thus $\vdash (((\alpha \implies \beta) \wedge (\beta \implies \gamma)) \Rightarrow (\alpha \implies \gamma))$. Note that it was not necessary to extend from **8** by using the ancestor **5** (as was done at **9**), because the left branch was already contradictory. □

Both proofs show that the wffs under consideration are true (and valid, in the second case). But consider the law of syllogism as applied to the following three predicates and two implications:

- $D(x)$: *x is a dog.*
- $P(x)$: *x is a pet.*
- $F(x)$: *x has fur.*

- $(D(x) \Rightarrow P(x))$: *If x is a dog, then x is a pet.*
- $(P(x) \Rightarrow F(x))$: *If x is a pet, then x has fur.*

The law of syllogism gives the conclusion $(D(x) \Rightarrow F(x))$: *If x is a dog, then x has fur.* However, there are dogs without fur. Though the argument is valid, a premise is false in our universe and thus the reasoning is not *sound*. This concept of soundness and its relationship with the tableau proof is explored more fully in the next chapter.

4.5 Examples

Recall that a thoerem in an axiomatic system is any true wff which can be obtained from the axioms via an accepted form of proof. This means any wff with a tableau proof whose hypothesis set is the axioms of the system is a theorem. This section will give several examples of such thoerems and their proofs in the axiomatic systems of Zermelo-Fraenkel set thoery, group theory, and Peano Arithmetic. This section will also show several examples of wffs which are valid in a particular model of a system via a tableau proof which includes both the system and the model axioms in its hypothesis set.

However, for convenience sake, the full set of axioms for each system are not always included in the hypothesis set; simply the ones needed for the proof. Similarly, steps in the proof may be condensed. For example, Γ_a may be written rather than Γ_b as part of a tree T

$\boxed{\Gamma_a}$

1: $\forall x \forall y \forall z(x = y \land y = z \Rightarrow x = z)$ EA3

|

2: $x = y \land y = z \Rightarrow x = z$ 1

$\boxed{\Gamma_b}$

1: $\forall x \forall y \forall z(x = y \land y = z \Rightarrow x = z)$ EA3

|

2: $\forall y \forall z(x = y \land y = z \Rightarrow x = z)$ 1

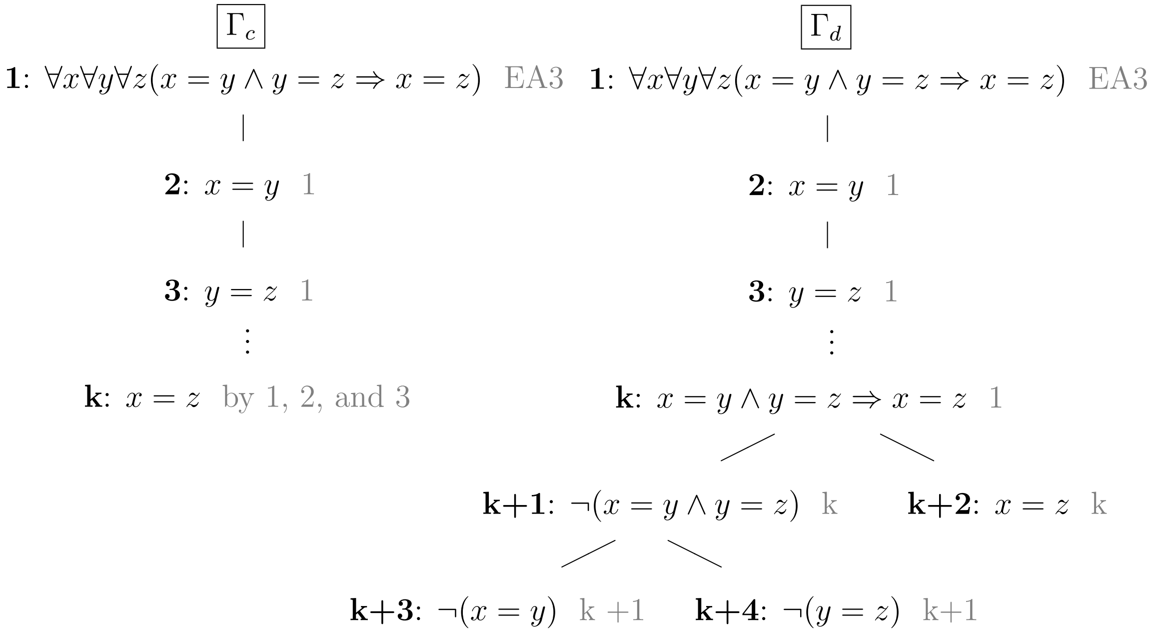
|

3: $\forall z(x = y \land y = z \Rightarrow x = z)$ 2

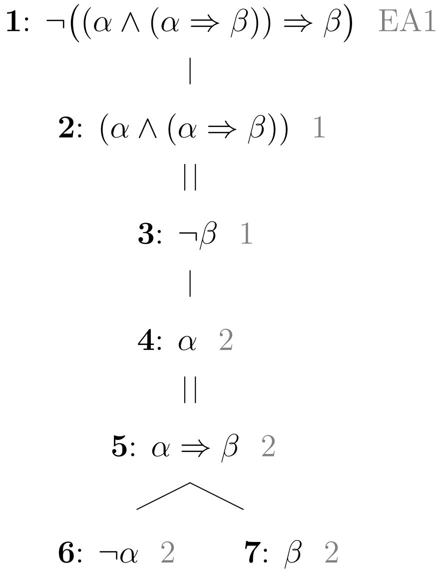
|

4: $x = y \land y = z \Rightarrow x = z$ 3

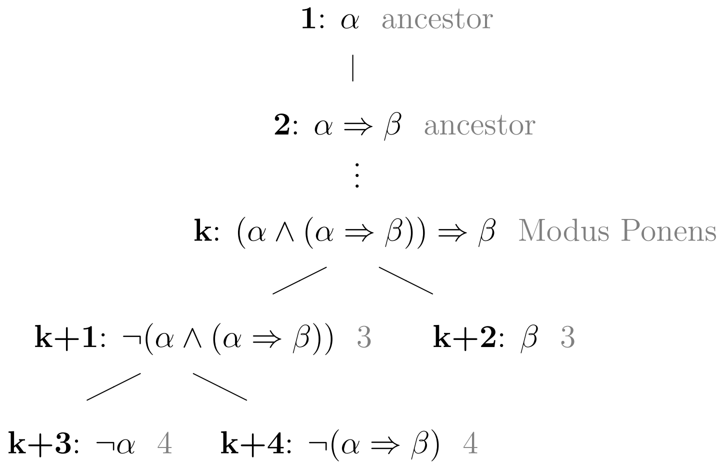
Similarly, if a tableau proof includes ancestor nodes $x = y$ and $y = z$, Γ_c may written for an extension to node k rather than the entire Γ_d (as the left two branches of Γ_d are contradictory).



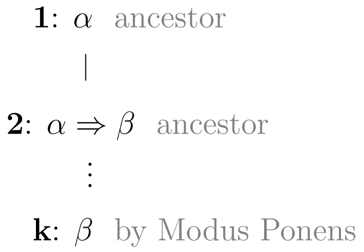
Also, any wffs which have been shown valid in first order logic may be used in tableau proofs of individual theorems for various systems. For example, modus ponens has the following tableau proof:



Thus, if a tableau contains ancestors α and $\alpha \Rightarrow \beta$, for an extension to k, the following may be added to said tableau:



As the left two branches are contradictory, a tableau proof with such ancestors is often simply given the extension β as follows:



Thus, keeping such shorthand in mind, it is possible to prove various theorems in ZF, GA, PA, and their associated models.

Example 4.27 Prove the empty set $\emptyset$ exists in the universe ZF, i.e. $\exists X \forall y(y \in X \wedge \neg(y = y))$ is a theorem.

1: $\neg\exists X\forall y(y \in X \wedge \neg(y = y))$ Negation of wff

|

2: $\forall y(y = y)$ EA1

|

3: $\neg\forall y(y \in X \wedge \neg(y = y))$ 1

|

4: $x \in X \wedge \neg(x = x)$ 3

|

5: $x \in X$ 4

||

6: $\neg(x = x)$ 4

|

7: $x = x$ 2

Note that at several nodes, a choice was made of which a or b was to be used for a substitution of the form $\beta(x//a)$ or $\beta(x//b)$ (where b did not occur at any ancestor of the bachelor node but a may have). Explicitly, these substitutions are listed here:

- **1** was of the form $\neg\exists$ and $a = X$ was chosen to form **3**.
- **3** was of the form $\neg\forall$ and $b = x$ was chosen to form **4**.
- **2** was of the form $\forall$ and $a = x$ was chosen to form **6**.

Including the axioms for a model in the hypothesis set of a tableau proof shows a wff is true in said model. However, note that such a proof does not show validity for the wff in every model.

Example 4.28 Prove that in the Grothendieck Universe $\mathbb{U}$ of ZF, if $x \in \mathbb{U}$ then $\{x\} \in \mathbb{U}$. That is, $x \in \mathbb{U} \Rightarrow \{x\} \in \mathbb{U}$ is true in this model of ZF.

1: $\neg(x \in \mathbb{U} \Rightarrow \{x\} \in \mathbb{U})$ Negation of wff

|

2: $\forall x \forall y(x \in \mathbb{U} \wedge y \in \mathbb{U} \Rightarrow \{x, y\} \in \mathbb{U})$ Second axiom of $\mathbb{U}$

|

3: $x \in \mathbb{U}$ 1

||

4: $\neg(\{x\} \in \mathbb{U})$ 1

|

5: $x \in \mathbb{U} \wedge x \in \mathbb{U} \Rightarrow \{x, x\} \in \mathbb{U})$ 2

6: $\neg(x \in \mathbb{U} \wedge x \in \mathbb{U})$ 5 **7**: $\{x, x\} \in \mathbb{U}$ 5

9: $\neg(x \in \mathbb{U})$ 6 **10**: $\neg(x \in \mathbb{U})$ 6 **8**: $\{x\} \in \mathbb{U}$ 7 (definition of set)

Thus $x \in \mathbb{U} \Rightarrow \{x\} \in \mathbb{U}$ is true in this model of ZF

□

Example 4.29 Prove inverses are unique in Group Theory, i.e. $\forall x \forall y \forall z \big(x, y, z \in G \Rightarrow (x * z = e = z * x \wedge y * z = e = z * y \Rightarrow x = y)\big)$ is valid.

1: $\neg\forall x\forall y\forall z\big(x, y, z \in G \Rightarrow (x * z = e = z * x \wedge y * z = e = z * y \Rightarrow x = y)\big)$ Negation

$|$

2: $\forall x\forall y\forall z(x, y, z \in G \Rightarrow (x * y) * z = x * (y * z))$ GA2

$|$

3: $\forall x(x \in G \Rightarrow e * x = x = x * e)$ GA3

$|$

4: $\forall x\forall y\forall z(x = y \wedge y = z \Rightarrow x = z)$ EA3

$|$

5: $\neg\big(a, b, c \in G \Rightarrow (a * c = e = c * a \wedge b * c = e = c * b \Rightarrow a = b)\big)$ 1

$|$

6: $a, b, c \in G$ 1

$||$

7: $\neg(a * c = e = c * a \wedge b * c = e = c * b \Rightarrow a = b)$ 1

$|$

8: $a * c = e = c * a \wedge b * c = e = c * b$ 7

$||$

9: $\neg(a = b)$ 7

$|$

10: $a * c = e = c * a$ 8

$||$

11: $b * c = e = c * b$ 8

$|$

12: $a = a * e$ 3

$|$

13: $a * e = a * (c * b)$ 11

$|$

14: $a * (c * b) = (a * c) * b$ 2

$|$

continued on next page

$$|$$

$$\textbf{15}\text{: } (a * c) * b = e * b \quad 10$$

$$|$$

$$\textbf{16}\text{: } e * b = b \quad 3$$

$$|$$

$$\textbf{17}\text{: } a = b \quad 4,\ 12\text{-}16$$

□

Example 4.30 Prove $(\mathbb{Z}_p, \cdot)$ is commutative in GA, i.e. $\forall x \forall y (x, y \in \mathbb{Z}_p \Rightarrow x \cdot y = y \cdot x)$. This tableau proof is actually very short, merely two lines, as the definition of multiplication requires that it respects products.

$$\textbf{1}\text{: } \neg\big(\forall x \forall y (x, y \in \mathbb{Z}_p \Rightarrow x \cdot y = y \cdot x)\big) \quad \text{Negation of wff}$$

$$|$$

$$\textbf{2}\text{: } \forall x \forall y (x, y \in \mathbb{Z}_p \Rightarrow x \cdot y = y \cdot x) \quad \text{definition of } \cdot$$

Example 4.31 Prove D_4 is not commutative in GA, i.e $\exists x \exists y (x, y \in D_4 \Rightarrow \neg(x \circ y = y \circ x))$, where elements of D_4 are assigned as in example 3.53.

1: $\neg\big(\exists x \exists y(x, y \in D_4 \Rightarrow \neg(x \circ y = y \circ x))\big)$ Negation of wff

|

2: $\neg(S_3 = S_1)$ By assignment in example 3.53

|

3: $\neg\big(\exists y(R_4, y \in D_4 \Rightarrow \neg(R_4 \circ y = y \circ R_4))\big)$ 1

|

4: $\neg(R_4, R_7 \in D_4 \Rightarrow \neg(R_4 \circ R_7 = R_7 \circ R_4))$ 3

|

5: $R_4, R_7 \in D_4$ 4

||

6: $\neg\neg(R_4 \circ R_7 = R_7 \circ R_4)$ 4

|

7: $(R_4 \circ R_7 = R_7 \circ R_4)$ 6

|

8: $(R_4 \circ R_7 = S_3)$ Definition of $\circ$

|

9: $(R_7 \circ R_4 = S_1)$ Definition of $\circ$

|

10: $(S_3 = S_1)$ By 7-9

Recall that a theorem proven by the axioms of a system must be valid in every model. Thus, when an wff (such as commutativity) is true in one model but false in another, the wff cannot be a thoerem of the axiomatic system of the models.

Example 4.32 Prove $s(0) + s(0) = s(s(0))$ in Peano Arithmetic.

1: $\neg(s(0) + s(0) = s(s(0)))$ Negation of wff

|

2: $\forall x \forall y \forall z(x = y \land y = z \Rightarrow x = z)$ EA3

|

3: $\forall x(x + 0 = x)$ PA4

|

4: $\forall x \forall y(x + s(y) = s(x + y))$ PA5

|

5: $s(0) + 0 = s(0)$ by 3

|

6: $s(0) + s(0) = s(s(0) + 0)$ by 4

|

7: $s(s(0) + 0) = s(s(0))$ by 3

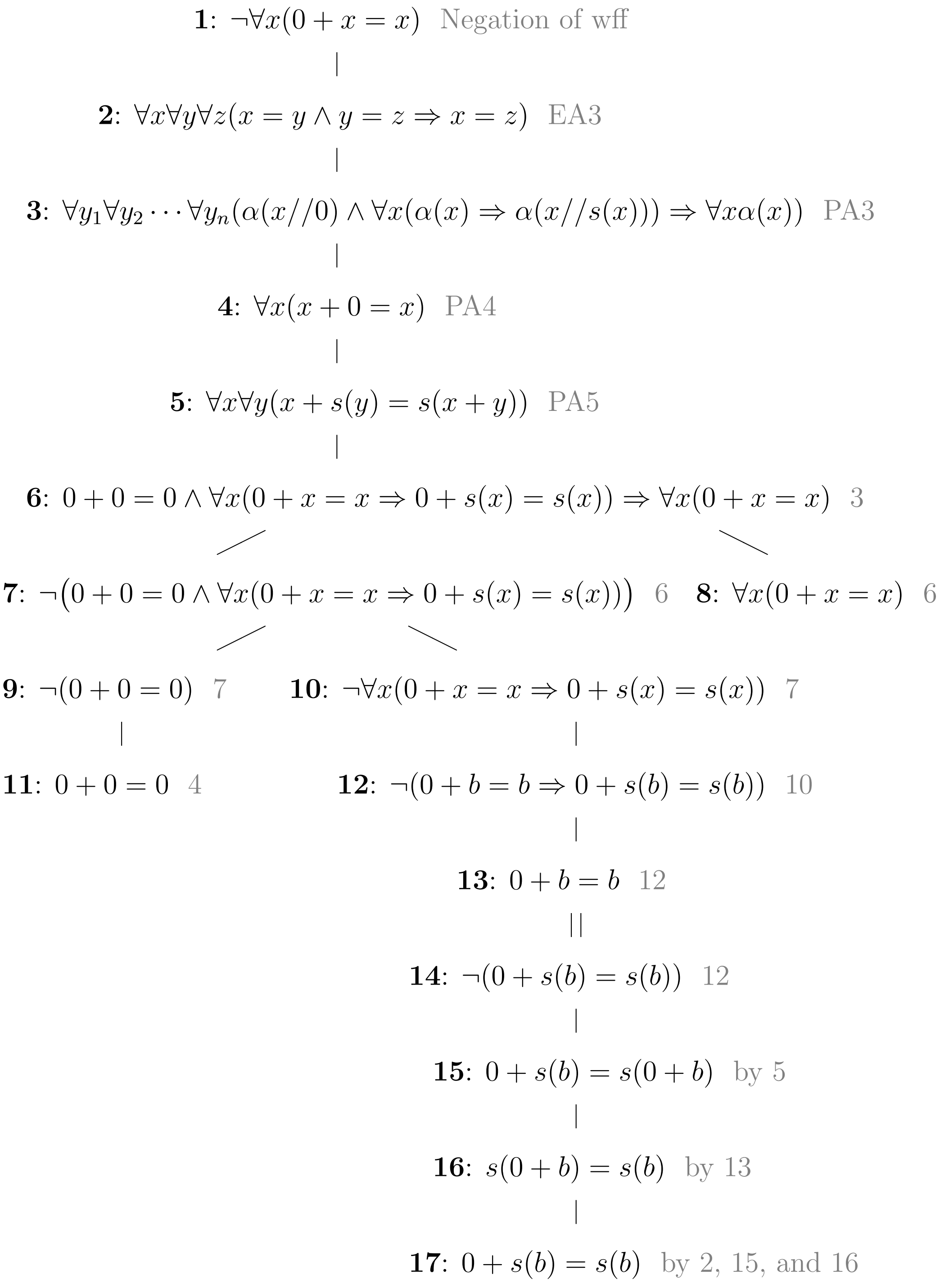
|

8: $s(0) + s(0) = s(s(0))$ by 2, 6, and 7

In the standard model of PA, this tableau proof shows $1 + 1 = 2$.

Example 4.33 Prove $\forall x(0 + x = x)$

This tableau proof will require PA3, where $\alpha(x) : (0 + x = x)$. If both $\alpha(0)$ and $\forall x(\alpha(x) \Rightarrow \alpha(s(x)))$ are shown to be true, then PA3, the Axiom of Induction, says $\forall x \alpha(x)$ is also true. To prove $\forall x(\alpha(x) \Rightarrow \alpha(s(x)))$, it suffices to show for an abitrary element of $\mathbb{N}$, the antecedent implies the consequent. This is illustrated by the following tableau proof.



1: $\neg\forall x(0 + x = x)$ Negation of wff

2: $\forall x \forall y \forall z(x = y \land y = z \Rightarrow x = z)$ EA3

3: $\forall y_1 \forall y_2 \cdots \forall y_n(\alpha(x//0) \land \forall x(\alpha(x) \Rightarrow \alpha(x//s(x))) \Rightarrow \forall x \alpha(x))$ PA3

4: $\forall x(x + 0 = x)$ PA4

5: $\forall x \forall y(x + s(y) = s(x + y))$ PA5

6: $0 + 0 = 0 \land \forall x(0 + x = x \Rightarrow 0 + s(x) = s(x)) \Rightarrow \forall x(0 + x = x)$ 3

7: $\neg\big(0 + 0 = 0 \land \forall x(0 + x = x \Rightarrow 0 + s(x) = s(x))\big)$ 6

8: $\forall x(0 + x = x)$ 6

9: $\neg(0 + 0 = 0)$ 7

10: $\neg\forall x(0 + x = x \Rightarrow 0 + s(x) = s(x))$ 7

11: $0 + 0 = 0$ 4

12: $\neg(0 + b = b \Rightarrow 0 + s(b) = s(b))$ 10

13: $0 + b = b$ 12

14: $\neg(0 + s(b) = s(b))$ 12

15: $0 + s(b) = s(0 + b)$ by 5

16: $s(0 + b) = s(b)$ by 13

17: $0 + s(b) = s(b)$ by 2, 15, and 16

This, in conjunction with axiom PA4, shows 0 is the additive identity. The multiplicative identity can be proven in a similar way, though it is necessary prove both $x \cdot s(0) = x$ and $s(0) \cdot x = x$

Example 4.34 Prove $s(o)$ is the multiplicative identity of PA, i.e. $\forall x(x \cdot s(0) = x) \wedge \forall x(s(0) \cdot x = x)$. This tableau proof is broken up into parts to manage its length. Not that $\alpha(x) : s(0) \cdot x = x$ when the Axiom of Induction is used.

$$\boxed{\mathrm{T}}$$

1: $\neg(\forall x(x \cdot s(0) = x) \wedge \forall x(s(0) \cdot x = x))$ Negation of wff

|

2: $\forall x \forall y \forall z(x = y \wedge y = z \Rightarrow x = z)$ EA3

|

3: $\forall y_1 \forall y_2 \cdots \forall y_n(\alpha(x//0) \wedge \forall x(\alpha(x) \Rightarrow \alpha(x//s(x))) \Rightarrow \forall x \alpha(x))$ PA3

|

4: $\forall x(x + 0 = x)$ PA4

|

5: $\forall x \forall y(x + s(y) = s(x + y))$ PA5

|

6: $\forall x(x \cdot 0 = 0)$ PA6

|

7: $\forall x \forall y(x \cdot s(y) = (x \cdot y) + x)$ PA7

|

8: $\forall x(0 + x = x)$ by example 4.33

/ \

$\boxed{\Gamma_a}$ $\boxed{\Gamma_b}$

$\boxed{\Gamma_b}$

10: $\neg(\forall x(s(0) \cdot x = x))$ 1

11: $s(0) \cdot 0 = 0 \wedge \forall x(s(0) \cdot x = x \Rightarrow s(0) \cdot s(x) = (x)) \Rightarrow \forall x(s(0) \cdot x = x)$ 3

12: $\neg\big(s(0) \cdot 0 = 0 \wedge \forall x(s(0) \cdot x = x \Rightarrow s(0) \cdot s(x) = (x))\big)$ 11 **13**: $\forall x(s(0) \cdot x = x)$ 11

14: $\neg(s(0) \cdot 0 = 0)$ 12 **15**: $\neg\forall x(s(0) \cdot x = x \Rightarrow s(0) \cdot s(x) = (x))$ 12

16: $s(0) \cdot 0 = 0$ 6

17: $\neg(s(0) \cdot b = b \Rightarrow s(0) \cdot s(b) = s(b))$ 15

18: $s(0) \cdot b = b$ 12

$||$

19: $\neg(s(0) \cdot s(b) = s(b))$ 12

20: $s(0) \cdot s(b) = (s(0) \cdot b) + s(0)$ 7

21: $(s(0) \cdot b) + s(0) = b + s(0)$ 18

22: $b + s(0) = s(b + 0)$ 5

23: $s(b + 0) = s(b)$ 4

24: $s(0) \cdot s(b) = s(b)$ 20-24, 2

$\boxed{\Gamma_a}$

9: $\neg(\forall x(x \cdot s(0) = x))$ 1

25: $\neg(b \cdot s(0) = b)$ 1

26: $b \cdot s(0) = (b \cdot 0) + b$ 7

27: $(b \cdot 0) + b = 0 + b$ 6

28: $0 + b = b$ 8

29: $b \cdot s(0) = b$ 26-28 and 2

Now a tableau proof of a wff α is a tableau confutation of $H \cup \{\neg\alpha\}$, that is, every branch of a finite tableau which contains $H \cup \{\neg\alpha\}$ as its root is contradictory. However, it has yet to be shown *why* a tableau confution of $\neg\alpha$ (where $H = \emptyset$) proves the validity of α. It relies on the reasoning that every branch of a finite tableau which contains $\neg\alpha$ in the root is either finished or contradictory. If a branch is finished, it (and therefore $\neg\alpha$) has a model by the Finished Set Theorem. If every branch is contradictory, there is no model of $\neg\alpha$ and thus α must be true in every model. In the next chapter, the Soundness Theorem will be proven which establishes that a tableau proof of a wff α implies α is valid.

Chapter 5

Soundness & Completeness

This chapter focuses on the relationship between proofs and validity. The Soundness Theorem will prove that a wff which has a tableau proof is valid. This will be accomplished by proving a series of smaller theorems. First, it will be shown that every finite tableau which has a finite or countable hypothesis set H that is modeled by M also has a branch which M models. The second step to prove the soundness theorem is a theorem which proves that if H has a tableau confutation, then H does not have a model. From this, the extended soundess theorem can be deduced, which states that if there is a tableau proof of a wff α from H, then α is a semantic consequence of H. The soundness theorem is simply the case where $H = \emptyset$.

The converse of soundess is completeness which states every valid wff has a tableau proof. To prove the Completeness Theorem it will be necessary to first prove every hypothesis set H, whether finite or countably infinite, has a tableau for which H is the root and whose every branch is either finished or contradictory and finite. This, along with the König Tree Theorem, will be used to show H has either a tableau confutation or a model. This will directly prove the Extended Completeness Theorem which gives the Completness Theorem when $H = \emptyset$.

5.1 Soundness

Soundness combines validity and semantics to judge the usefullness and applicability of methods of logical reasoning. Explicitly, soundness is the property which says any wff which has a proof of validity is also true. The previous chapter claimed a tableau proof of a wff α established the validity of α. The Soundness Theorem proves this claim. Recall that a tableau proof is a tableau confutation of $H \cup \{\neg\alpha\}$ where α is the wff under consideration and H is the hypothesis set. When $H = \emptyset$, then a tableau confutation of $\emptyset \cup \{\neg\alpha\} = \{\neg\alpha\}$ is a proof that α is valid. By definition, a wff is valid only if it is true in every model, thus the property of soundness can be stated as the following thoeroem:

Theorem 5.1 The Soundness Theorem: *If a wff has a tableau proof, then it is valid.*

The proof for the soundness theorem will be approached using a series of theorems that can be proven simultaneously for sentential and first order logic. First, it will be shown that every finite tableau which has a finite or countable hypothesis set H that is modeled by M also has a branch which M models. This is expressed by the following theorem:

Theorem 5.2 *If T is a tableau with finite or countable hypothesis set H and $M \models H$, then there is a branch Γ of T such that $M \models \Gamma$.*

However, in first order logic, this occurs in a universe U, and the wffs may have free variables or parameter symbols x_1, x_2, $\cdots$, x_n, that are not in

U. Thus it is necessary to replace such individuals with suitable elements a_1, a_2, $\cdots$, a_n, from U. Such a replacement in a wff α is called a valuation, and will either be denoted by $\alpha(x_1, x_2, \cdots, x_n//a_1, a_2, \cdots, a_n)$ or $\alpha(v)$ where v is the substitution $(x_1, x_2, \cdots, x_n//a_1, a_2, \cdots, a_n)$. Similarly, the notation $\Gamma(v)$, where Γ is a set, indicates every wff α in Γ is of the form $\alpha(v)$. Thus in first order logic, theorem 5.2 actually says $M \models \Gamma(v)$ for some v. Hypothesis sets H from first order logic will also be considered to only contain wffs with no free variables or parameters.

The proof for theroem 5.2 is the longest and most complicated in this section. It uses induction and case analysis on the formation of the tableau T, the branch Γ or, in first order logic, $\Gamma(v)$, and the valuation v. The theorem is even broken into two parts, a and b, the first of which proves the theorem in sentential logic and the second in first order. Thus, before even attempting the proof, it is helpful to first consider a small example tableau.

Example 5.3 Let T be a first order logic tableau in a universe U with the finite hypothesis set $\{\exists x\big(\alpha(x) \Rightarrow \beta(x)\big)\}$, modeled by M. Prove there is a branch Γ in T which M also models, i.e. $M \models \Gamma(v)$ for some valuation v. Now T is T_n for some finite tableau chain T_0, T_1, $\cdots$, T_n. Thus the branch which M models can be found by examining the formation of the various branches in the tableau chain as well as any valuations $(x//y)$ which occur. Now T_0 is the hypothesis set, so there is only one possible branch Γ_0 which also contains the hypothesis set.

$$\boxed{\mathrm{T}_0 = \Gamma_0}$$

1: $\exists x\big(\alpha(x) \Rightarrow \beta(x)\big)$ Hypothesis

There are no free variables or parameters to consider, so the valuation corresponding to Γ_0 is the empty set, that is, $v_0 = \emptyset$. Obviously $M \models \Gamma_0$ as M models the hypthesis set. By proposition 3.44, $M \models \exists x\big(\alpha(x) \Rightarrow \beta(x)\big)$ implies $M \models \big(\alpha(x//b) \Rightarrow \beta(x//b)\big)$ for some $b \in U$. This means when T_1 is obtained from T_0 by extending at **1**, T_1 and Γ_1 are both formed by adding the child $\alpha(x//y) \Rightarrow \beta(x//y)$ and creating the valuation $v_1 = (y//b)$.

$$\boxed{\mathrm{T}_1 = \Gamma_1}$$

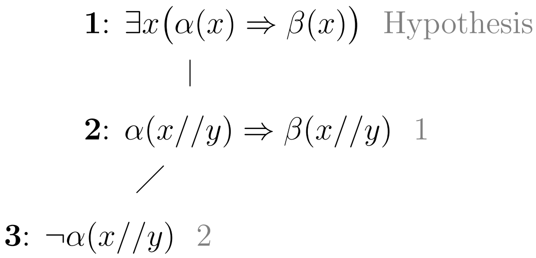
1: $\exists x\big(\alpha(x) \Rightarrow \beta(x)\big)$ Hypothesis

|

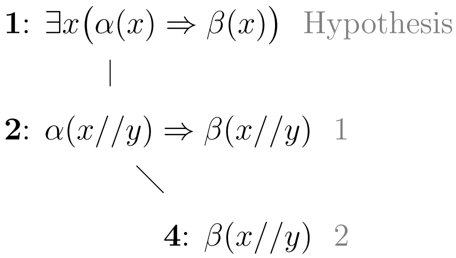
2: $\alpha(x//y) \Rightarrow \beta(x//y)$ 1

Because $M \models \big(\alpha(x//b) \Rightarrow \beta(x//b)\big)$, $M \models \Gamma_1(v_1)$. Now there are two ways to extend T_1 using **2**, so Γ_2 could be formed by either the addition of the node corresponding to the wff $\neg\alpha(x//b)$ or $\beta(x//b)$, exclusively. As there are no new free variables or parameters to consider, $v_2 = v_1$.

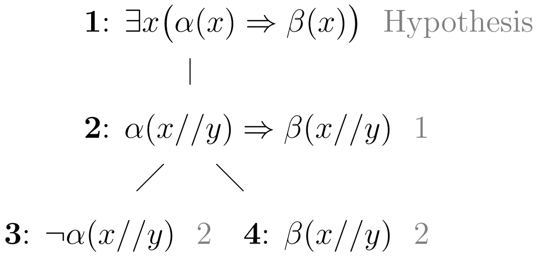
$\boxed{\Gamma_2 : M \models \neg\alpha}$



$\boxed{\Gamma_2 : M \models \beta}$



$\boxed{\mathrm{T}_2 = T}$



Now T_2 is finished, so T_2=T. By proposition 2.48, either $M \models \neg\alpha(x//b)$ or $M \models \neg\beta(x//b)$, or both. Γ_2 is chosen based on which wff is modeled by M. Thus M models every wff in the branch $\Gamma_2(v_2) = \Gamma_2(y//b)$ of T, so $M \models \Gamma_2(y//b)$. $\square$

The proof for the theorem will proceed in much the same way as this example. A chain of branches corresponding to the tableau chain will be constructed. Induction will prove each one is modeled by M. In the proof for first order logic, the valuations will be constructed as well.

Theorem 5.2a *If T is a finite sentential tableau with finite or countable hypothesis set H which is modeled by M, then there is a branch Γ of T such that $M \models \Gamma$.*

Proof: To prove this theorem, it is necessary to find at least one branch Γ of T where every wff which occurs on Γ is true in M. From definition 4.22 it is known T is T_n for some finite tableau chain T_0, T_1, $\cdots$, T_n. Thus it is possible to construct a chain of branches Γ_0 , Γ_1, $\cdots$, Γ_n where each branch is formed from the one before it and each Γ_k is a branch of T_k. Through induction, every wff on those branches are proven to be modeled by M. The last branch, Γ_n, will satisfy the theorem.

Consider the base case Γ_0. Now T_0 is the root node and contains only those wffs from H, so a branch Γ_0 of T_0, can be defined as the set containing the root node. As M models every wff in T_0, it models every wff on the root node and thus every wff in Γ_0.

Assume M models the branch Γ_k of T_k, for some $k < n$. Now T_{k+1} is obtained from T_k by extending at some bachelor node. If this is done other than at the bachelor node of Γ_k, Γ_{k+1} is defined to be Γ_k. As M models Γ_k, M models Γ_{k+1}. If T_{k+1} is found by extending at the bachelor node of Γ_k, this is done by applying a tableau extension rule to a wff of T_k There are nine ways in which this can happen, one for each of the tableau extension rules. Thus if γ is the ancestor wff used, Γ_{k+1} is defined to be obtained from Γ_k in the following ways:

- Case 1: $\gamma = \neg\neg\alpha$. Form Γ_{k+1} by adding a child β to Γ_k.
- Case 2: $\gamma = \alpha \wedge \beta$. Form Γ_{k+1} by adding a child α and grandchild β to Γ_k.
- Case 3: : $\gamma = \neg(\alpha \wedge \beta)$. Form Γ_{k+1} by adding a child $\neg\alpha$ or a child $\neg\beta$ to Γ_k, whichever M models.
- Case 4 : $\gamma = \alpha \vee \beta$. Form Γ_{k+1} by adding a child α or a child β to Γ_k,

whichever M models.

- Case 5: $\gamma = \neg(\alpha \vee \beta)$. Form Γ_{k+1} by adding a child $\neg\alpha$ and grandchild $\neg\beta$ to Γ_k.
- Case 6 : $\gamma = \alpha \Rightarrow \beta$. Form Γ_{k+1} by adding a child $\neg\alpha$ or a child β to Γ_k, whichever M models.
- Case 7 : $\gamma = \neg(\alpha \Rightarrow \beta)$. Form Γ_{k+1} is formed by adding a child α and a grandchild $\neg\beta$ to Γ_k.
- Case 8 : $\gamma = \alpha \Leftrightarrow \beta$. Form Γ_{k+1} by adding a child α and grandchild β or by adding a child $\neg\alpha$ and grandchild $\neg\beta$, whichever two M models.
- Case 9: $\gamma = \neg(\alpha \Leftrightarrow \beta)$. Form Γ_{k+1} by adding a child α and grandchild $\neg\beta$ or by adding a child $\neg\alpha$ and grandchild β, whichever two M models.

In these cases, “or” is exclusive (if it was inclusive, Γ_{k+1} would become two branches). By proposition 2.48, in every case of above, as M models the ancestor γ, M models every wff added. Thus, as M is assumed to model Γ_k, M models Γ_{k+1}. By construction, Γ_{k+1} is a branch of T_{k+1}. Therefore, by induction, M models a branch Γ_n of $\mathrm{T}_n = \mathrm{T}$ and the threorem is proven.

Theorem 5.2b *If T is a finite first order logic tableau with a finite or countable hypothesis set H which is modeled by M, then there is a branch Γ of T such than $M \models \Gamma(v)$ for some valuation v.*

Proof: This theorem is very similar to that of 5.2**a** except the wffs γ which occur along every branch Γ_k must be modeled by M as $\gamma(v_k)$ for some valuation $v_k = (x_1,\, x_2,\, \cdots,\, x_l \;//\; a_1,\, a_2,\, \cdots,\, a_l)$ whose substituted variables and parameters are free in Γ_k. Both the branches Γ_k and the valuations v_k must be inductively defined.

When $k = 0$, Γ_0 is the root node containing only those wffs from H. As wffs from H in first order logic are considered to have no free variables or parameters, v_0 is considered to be the empty set.

Assume M models $\Gamma_k(v_k)$ for some $k < n$. If T_{k+1} is obtained from T_k by extending at some node other than the bachelor node of Γ_k, $\Gamma_{k+1} = \Gamma_k$ and $v_{k+1} = v_k$. Then M models $\Gamma_{k+1}(v_{k+1})$ as M models $\Gamma_k(v_k)$. If T_{k+1} is obtained from T_k by extending at the bachelor node of Γ_k, this is done by applying a tableau extension rule to an ancestor γ of Γ_k. If it is done in one of the nine ways given by the proof for 5.2a then $v_k = v_{k+1}$ as there are no new free variables or parameters to consider for the valuation. In the remaining four scenarios, Γ_{k+1} and v_{k+1} can be obtained in the following way:

- Case 1: $\gamma = \forall x \alpha$. Form Γ_{k+1} by adding the child $\alpha(x//y)$ where y is a free variable or parameter. If y has already been substituted in v_k, define $v_{k+1} = v_k$. If not, define

  $$v_{k+1} = (x_1, x_2, \cdots, x_l, y//a_1, a_2, \cdots, a_l, b)$$

  where b is in U. By proposition 3.44, as M models the ancestor $\gamma(v_k)$, M models $\alpha(v_{k+1})$.

- Case 2: $\gamma = \neg\forall x \alpha$. Form Γ_{k+1} by adding the child $\neg\alpha(x//y)$ where y is a free variable or parameter that does not occur in any ancestor of γ. Then y is not substituted in the valuation so define

  $$v_{k+1} = (x_1, x_2, \cdots, x_l, y//a_1, a_2, \cdots, a_l, b)$$

  where b is in U and chosen such that M models $\neg\alpha(v_{k+1})$. Such a b is gauranteed by proposition 3.44.

- Case 3: $\gamma = \exists x\alpha$ .Form Γ_{k+1} by adding the child $\alpha(x//y)$ where y is a free variable or parameter that does not occur in any ancestor of γ. Then y is not substituted in the valuation so define

  $$v_{k+1} = (x_1, x_2, \cdots, x_l, y//a_1, a_2, \cdots, a_l, b)$$

  where b is in U and chosen such that M models $\alpha(v_{k+1})$. Such a b is gauranteed by proposition 3.44.

- Case 4: $\gamma = \neg\exists x\alpha$. Form Γ_{k+1} by adding the child $\neg\alpha(x//y)$ where y is a free variable or parameter. If y has already been substituted in v_k, define $v_{k+1} = v_k$. If not, define

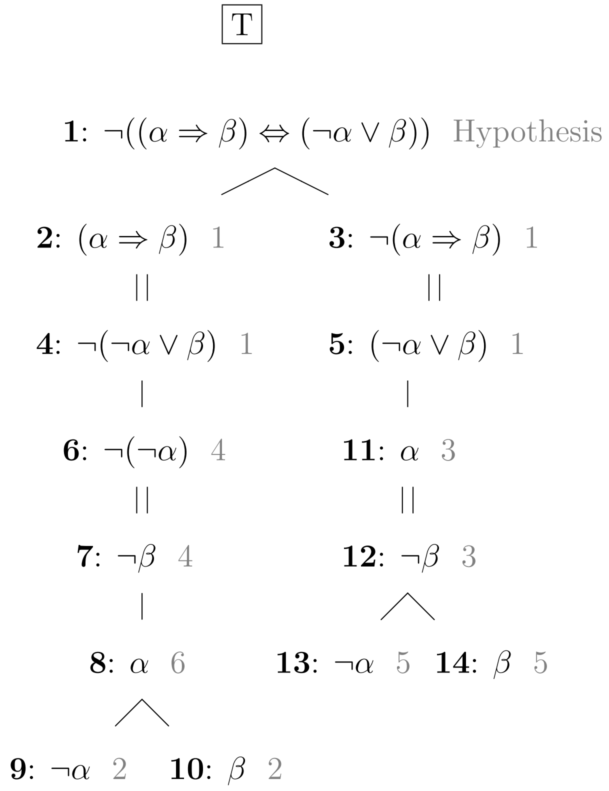
  $$v_{k+1} = (x_1, x_2, \cdots, x_l, y//a_1, a_2, \cdots, a_l, b)$$

  where b is in U. By proposition 3.44, as M models the ancestor $\gamma(v_k)$, M models $\neg\alpha(v_{k+1})$.

Hence M models $\Gamma_{k+1}(v_{k+1})$ for every $k < n$ and thus models a branch $\Gamma_n(v_n)$ of T_n =T and the theorem is proven. ▮

The second step to prove the soundness theorem is a proof that if H has a tableau confutation, then H does not have a model (recall that a tableau confutation of H is a finite tableau with root H where every branch is contradictory). This is a reversal from the previous theorem which proved a that a model of the hypothesis set also modeled a branch of its corresponding tableau. This lack of a model is easy to see in a small example of sentential logic where every model can be considered.

Example 5.4 Consider the wff $\neg((\alpha \Rightarrow \beta) \Leftrightarrow (\neg\alpha \vee \beta))$. This has the following tableau confutation.



However, as this wff is of sentential logic, it can also be viewed in truth table form.

α	β	$\neg\alpha$	$\alpha \Rightarrow \beta$	$\neg\alpha \vee \beta$	$((\alpha \Rightarrow \beta) \Leftrightarrow (\neg\alpha \vee \beta))$	$\neg((\alpha \Rightarrow \beta) \Leftrightarrow (\neg\alpha \vee \beta))$
T	T	F	T	T	T	F
T	F	F	F	F	T	F
F	T	T	T	T	T	F
F	F	T	T	T	T	F

In this form, it is easy to see that in every possible model, the wff will be false. This is reasonable, as by the previous theorem, if the wff did have a model M it would be a model for some branch of the tableau confutation. But every branch is contradictory, including a wff and its negation, so cannot be modeled. $\square$

The proof for this theorem (explicitly given below) uses the same reasoning, showing by contradiction and theorem 5.2 that no model can exist for a hypothesis set with a tableau confutation.

Theorem 5.5 *If H is a finite or countable set of wffs with a tableau confutation, then H has no model.*

Proof: Let H be a finite or countable set of wffs and T a tableau confutation of H. Assume M models H. Then by theorem 5.2 (a and b), M models some branch Γ of T ($\Gamma(v)$ for some valuation v in first order logic). But T has a tableau confutation so every branch, including Γ ($\Gamma(v)$), contains a wff and its negation. But M cannot model both a wff and its negation, thus there can be no model of H. ▮

Note that M cannot model both a wff α and its negation because it would require two different n-ary relations be assigned to the same wff, an impossibility in a model. Recall that a wff α is a semantic consequence of a set of wffs H if every model of H is a model of α. The extended soundness theorem integrates the tableau with semantic consequence and shows that a tableau proof of α from H proves α is a semantic consequence of H.

Theorem 5.6 The Extended Soundness Theorem: *If H is a finite or countable set of wffs which give a tableau proof of a wff α, then α is a semantic conseqence of H. That is, if $H \vdash \alpha$ then $H \models \alpha$.*

Proof: Let H be a finite or countable set of wffs which give a tableau proof of a wff α. Then there is a tableau confutation of $H \cup \{\neg\alpha\}$. By theroem 5.5, there is no model M of $H \cup \{\neg\alpha\}$, so no model M of H is a model of $\neg\alpha$. Thus, if M does model H, it must model α (rather than $\neg\alpha$). Therefore, by definition 4.1, $H \models \alpha$. ∎

When H is the empty set, then a tableau proof shows α is valid. Then we have theorem 5.1, where $\vdash \alpha$ implies $\emptyset \models \alpha$. There have been many examples of tableau proofs showing validity in this thesis, including

- The tree diagram at the beginning of section 4.4 which proved the validity of modus ponens.
- T_8 of the finite tableau chain of example 4.21 which proved the validity of a DeMorgan's law.
- The second tableau proof of example 4.26 which proved the law of syllogism.
- T_2 of example 5.3 which proved the validity of $\exists x(\alpha(x) \Rightarrow \beta(x))$
- The tableau proof of example 5.4 which proved the validity of $(\alpha \Rightarrow \beta) \Leftrightarrow (\neg\alpha \vee \beta)$.
- Examples of section 4.5.

5.2 Completeness

Completeness is simply the converse of soundness. While soundness claims every wff with a tableau proof is valid, completeness states every valid wff has a tableau proof.

Theorem 5.7 The Completeness Theorem: *If a wff is valid, then it has a tableau proof.*

The Completeness Theorem will be proven for both sentential and first order logic in a series of steps and theorems. Theorems 5.8, 5.9, and 5.10 will prove the first step towards the completeness theorem: proving every finite and countably infinite hypothesis set of both sentential and first order logic has a tableau for whigh H is the root.

Theorem 5.8 *Every finite hypothesis set H of sentential logic has a finite sentential logic tableau T.*

Proof: Let H be a finite hypothesis set of sentential logic. Through induction, it is possible to construct a finite sentential logic tableau chain T_0, T_1, $\cdots$, T_k, $\cdots$, T_n whose last labeled tree T_n is a finite tableau for H. Let $u(\mathrm{T}_k)$ be the length of the longest unused wff in a labelled tree T_k where $u(\mathrm{T}_k)$=0 only if there are no unused wffs in T_k.

Now T_0 is the labelled tree containing the hypothesis set. If $u(\mathrm{T}_0)$=0, then there are no unused wffs and T_0=T_n is a finite tableau. If $u(\mathrm{T}_0)\neq 0$, then form T_1 by extending at the bachelor node of T_0 using a tableau extension rule with the longest unused wff of T_0 (if several have the same longest length, simply pick one). Then T_1 is formed according to definition 4.18.

Assume T_0, T_1, $\cdots$, T_k is the first part of a tableau chain satisfying definition 4.18. If $u(\mathrm{T}_k)=0$, then there are no unused wffs and $\mathrm{T}_k=\mathrm{T}_n$ is a finite tableau. If $u(\mathrm{T}_k)\neq 0$, then extend to T_{k+1} by using the longest unused wff α of T_k on a noncontradictory branch that includes α. Then T_{k+1} is formed according to definition 4.18.

As the hypothesis set is finite, eventually $u(\mathrm{T}_n)=0$ for some n. Thus, by induction, T_0, T_1, $\cdots$, T_k, $\cdots$, T_n is a tableau chain and thus T_n is a finite tableau with root H. ∎

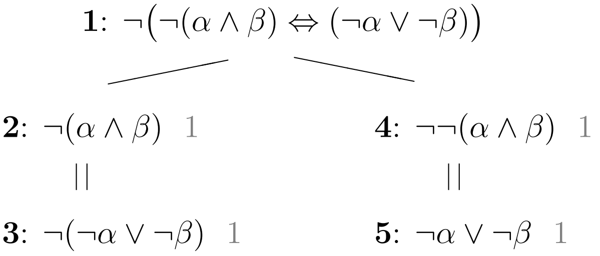
It is important to note that, as T is a finite tableau, it follows from definitions 4.18 and 4.22 that each of its branches are either finished or contradictory (and finite). Essentially, the previous proof explicitly outlines an empirical way to create a tableau chain from a finite hypothesis set using the tableau extension rules. However, it does not necessarily give the quickest way.

For instance, consider the tableau chain given in example 4.21. The hypothesis set was simply the wff $\neg\big(\neg(\alpha\wedge\beta)\Leftrightarrow(\neg\alpha\vee\neg\beta)\big)$. Following the outline of the proof, the tableau chain goes as follows:

$$\boxed{\mathrm{T}_0}$$

$$\mathbf{1}\text{: } \neg\big(\neg(\alpha\wedge\beta)\Leftrightarrow(\neg\alpha\vee\neg\beta)\big)$$

$$\boxed{\mathrm{T}_1}$$



$$\mathbf{1}\text{: } \neg\big(\neg(\alpha\wedge\beta)\Leftrightarrow(\neg\alpha\vee\neg\beta)\big)$$

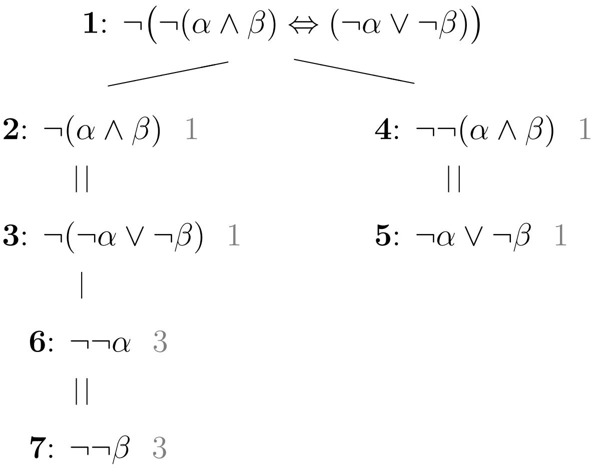
$$\mathbf{2}\text{: } \neg(\alpha\wedge\beta) \;\; 1 \qquad\qquad \mathbf{4}\text{: } \neg\neg(\alpha\wedge\beta) \;\; 1$$

$$||\qquad\qquad\qquad\qquad ||$$

$$\mathbf{3}\text{: } \neg(\neg\alpha\vee\neg\beta) \;\; 1 \qquad\qquad \mathbf{5}\text{: } \neg\alpha\vee\neg\beta \;\; 1$$

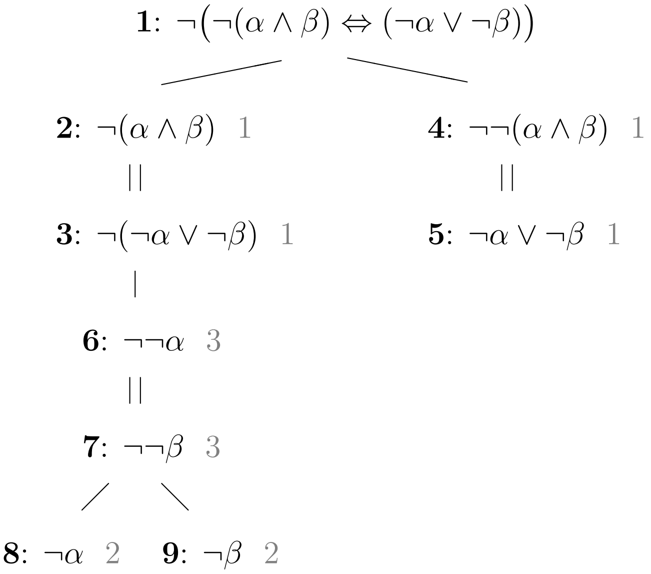
At this point, T_1 has unused wffs of length (in order of node numbering) 6, 8, 7, and 5. Thus T_2 will be formed by using the node **3** (as was done in example 4.21).

$\boxed{\mathrm{T}_2}$



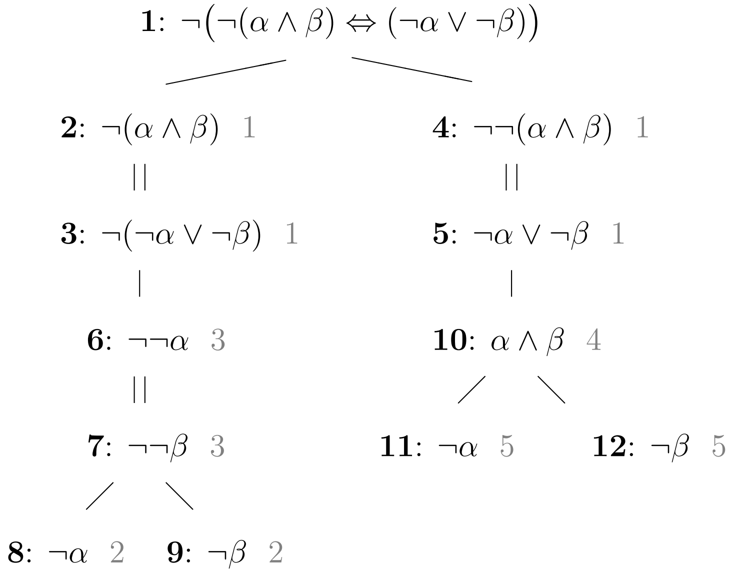
However, now node **2** rather than **4** will used to form T_3.

$\boxed{\mathrm{T}_3}$



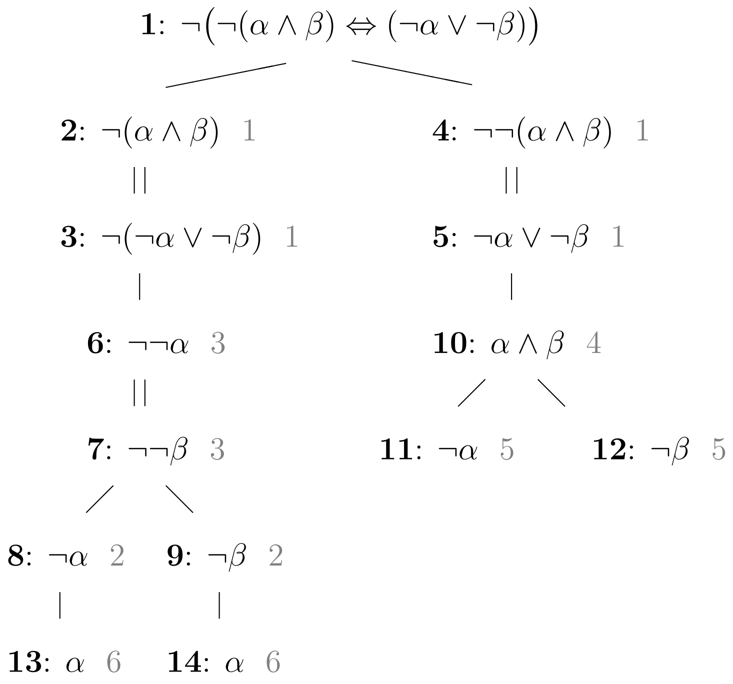
As this creates a branch, the remaining ancestor nodes **4** and **5** will have to be used twice on each new branch, rather than once as in the previous example. The nodes **4** and **5** are each extended from once to form T_4 and then T_5

$\boxed{T_5}$

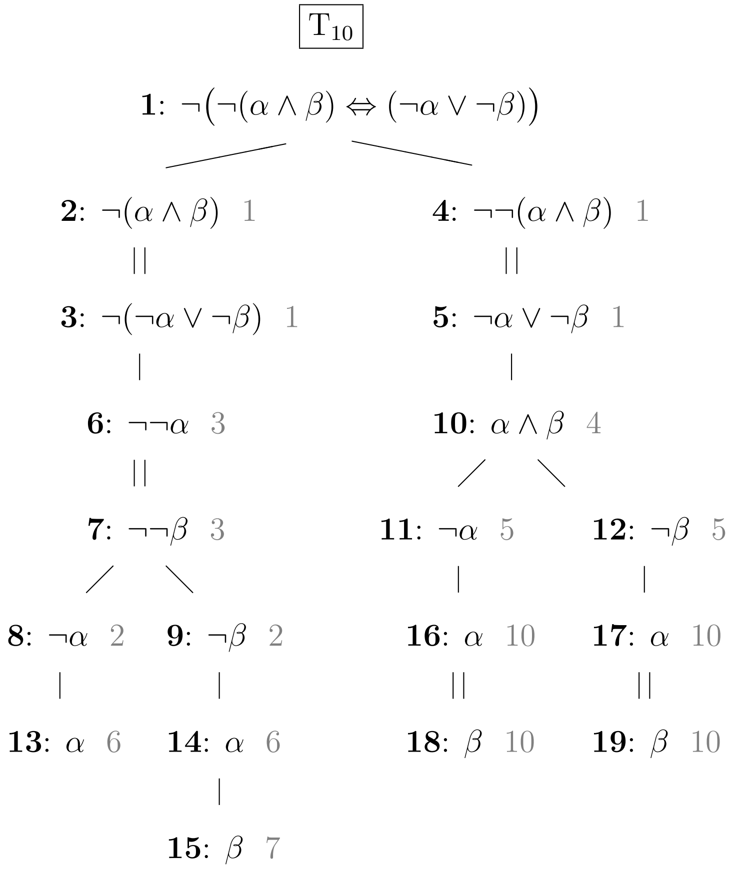


At this point, there are three remaining nodes which are unused and all have the same length. Thus they can be chosen arbitrarily, in this case by node numbering (**6** then **7** then **10**). Two extensions using **6** in each of its branches yields two more trees, T_6 and T_7.

$\boxed{\mathrm{T}_7}$



Now node **7** is used to form T_8. As the far left branch is contradictory, it is not necessary to extend through the bachelor node **13**, so the extension only occurs through **14**. An additional extension using node **10** can occur twice through each branch it is an ancestor of, giving T_{10}



Note that not only are there three more nodes, it took two more labeled trees to complete the tableau chain. The next theorem is very similar to the previous one, proving a countably infinite hypothesis set can also have a tableau which has many of the properties of a finite tableau.

Theorem 5.9 *Every countably infinite hypothesis set H of sentential logic has a tableau T such that every branch is either finished or contradictory and finite.*

Proof: Let H be a countably infinite set of hypothesis $H = \{\alpha_1, \alpha_2, \cdots, \alpha_k, \cdots\}$ and H_k be a finite subset of the first k hypothesis of H, $H_k = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$. By theorem 5.8, every H_k has a corresponding finite tableau T_k. Thus it is possible to use the theorem countably many times and get a sequence of finite tableaus T_0, T_1, $\cdots$, T_k, $\cdots$, such that:

- T_0 contains only a root node,
- For each $k > 0$, T_k is a finite tableau for H_k, and
- For each $k > 0$, T_{k+1} is obtained from T_k in acccordance with definition 4.18.

Let T be the union T$= \cup_{k=0}^{\infty}\mathrm{T}_k$ and Γ a branch of T. If Γ is contradictory, then it contains both a wff α and its negation $\neg\alpha$. Then there exists some k such that both α and $\neg\alpha$ are in T_K. Then $\Gamma \cap \mathrm{T}_k$ is already a contradictory branch of T_k. But by definition 4.18, the contradictory branch $\Gamma \cap \mathrm{T}_k$ is never extended after T_k is obtained, so $\Gamma = \Gamma \cap \mathrm{T}_k$ and thus is finite. If Γ is not contradictory, then, as it was constructed in accordance with definition 4.18, it must be finished (recall the definition of a finished set, 4.8).

Hence T is a tableau such that every branch is either finished or contradictory. ▮

Both of these last two theorems were applicable only in sentential logic, but they can also be proven for first order logic in the form of the next theorem. Note that, as in the previous chapter, any hypothesis set H of first order logic will be assumed to have no free variables or parameters.

Theorem 5.10 *Let U be a finite or countably infinite set of parameter symbols and H be a finite or countably infinite set of first order logic wffs. Then there is a tableau T on U with root H such that no free variables occur on T and every branch of T is either finished or contradictory and finite.*

Proof: Let U be a countably infinite set of parameter symbols $U = \{u_1, u_2, \cdots, u_k, \cdots\}$, H a finite or countably infinite set of first order logic wffs, $H = \{\alpha_1, \alpha_2, \cdots, \alpha_k, \cdots\}$, and H_k be the set of the first k elements of

H, $H_k = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$, where, if H is finite, $H_n = H$ for $n \geq k$. Then it is possible to define a sequence of finite tableaus T_0, T_1, $\cdots$, T_k, $\cdots$ such that:

- T_0 contains only a root node,
- For each $k > 0$, T_k is a finite tableau with root H_k,
- For each wff α on a non-contradictory branch Γ of T_{k+1} such that $\alpha \in \mathrm{T}_k$ the following hold:
  - If $\alpha = \forall x \beta$ then $\beta(x//u_i)$ is also on Γ for every $i \in \{1, 2, \cdots, k+1\}$.
  - If $\alpha = \neg\exists x \beta$ then $\neg\beta(x//u_i)$ is also on Γ for every $i \in \{1, 2, \cdots, k+1\}$.
  - If α is of any other form, α is used at least once along Γ according to the tableau extension rules of definition 4.19.

This means if α is of the form $\neg\forall x \beta$ or $\exists x \beta$, then $\neg\beta(x//u_i)$ or $\beta(x//u_i)$,respectively, is on Γ for at least one $i \in \{1, 2, \cdots, k, \cdots\}$. Similarly, if α has the form $\beta_1 \vee \beta_2$, $\neg(\beta_1 \wedge \beta_2)$, or $\beta_1 \Rightarrow \beta_2$, exactly one child is on Γ.

Thus T_{k+1} is constructed from T_k in finitely many stages (note that if the wffs of form $\forall$ and $\neg\exists$ had not been restricted to $i \in \{1, 2, \cdots, k+1\}$ this would not be true). Consider the tableau $\mathrm{T}{=}\cup_{k=0}^{\infty}\mathrm{T}_k$. It is necessary to prove that every branch of T is either finished or contradictory.

If Γ is contradictory, then it contains both a wff α and its negation $\neg\alpha$. Then there exists some k such that both α and $\neg\alpha$ are in T_K. Then $\Gamma{\cap}\mathrm{T}_k$ is already a contradictory branch of T_k. But by definition 4.19, the contradictory branch $\Gamma{\cap}\mathrm{T}_k$ is never extended after T_k is obtained, so $\Gamma = \Gamma{\cap}\mathrm{T}_k$ and thus is finite.

If Γ is not contradictory then it is necessary to show its wffs are a finished set. Let α be a wff on Γ. Then for some k, α is in T_k. Now $\Gamma \cap \mathrm{T}_{k+1}$ is a branch of T_{k+1}, so by construction α has been used in $\Gamma \cap \mathrm{T}_{k+1}$ and thus Γ. Thus, the following is true:

- If α is of the form $\forall x \beta$ or $\neg \exists x \beta$ then for every $n > k$ and $i \leq n$, $\beta(x//u_i)$ or $\neg \beta(x//u_i)$, respectively, is on $\Gamma \cap \mathrm{T}_n$.
- If α is of the form $\neg \forall x \beta$ or $\exists x \beta$, then $\beta(x//u_i)$ is in Γ for every $i \in \{1, 2, \cdots, k, \cdots\}$.
- If α has the form $\beta_1 \vee \beta_2$, $\neg(\beta_1 \wedge \beta_2)$, or $\beta_1 \Rightarrow \beta_2$, then exactly one child is in Γ.
- If α is of the form $\neg\neg\beta$, then β is in Γ.
- If α is of the form $\beta_1 \wedge \beta_2$, $\neg(\beta_1 \vee \beta_2)$, or $\neg(\beta_1 \Rightarrow \beta_2$ then both the child and grandchild are in Γ.
- If α is of the form $\beta_1 \Leftrightarrow \beta_2$ or $\neg(\beta_1 \Leftrightarrow \beta_2)$ then one child and the associated grandchild are in Γ.

Thus, by definition 4.8, Γ is a finished branch. Thus T is a tableau with hypothesis set H whose branches are either finished or contradictory and finite. ∎

Thus it is now known every finite or countably infinite hypothesis set H of sentential logic or first order logic has a tableau whose every branch is either finished or contradictory and finite.

Recall that it has been proven that an implication $\alpha \Rightarrow \beta$ and its contrapositive $\neg\beta \Rightarrow \neg\alpha$ are equivalent (see example 4.24). The König Tree

Theorem proves a result of the form $\alpha \Rightarrow \beta$, but only its contrapositive is used in later theorems.

Theorem 5.11 The König Tree Theorem: *If a tree T has infinitely many nodes and each node has a finite number of children, then there is a branch* Γ *of T such that* Γ *is infinite.*

Proof: This proof will construct by induction an infinite branch of T consisiting of an infinite sequence of nodes $t_0, t_1, t_2, \cdots$ such that t_0 is the root node, t_{n+1} is a child of t_n, and each t_n has an infinite number of descendents (children, grandchildren, great-grandchildren, etc).

Let T be a tree with infinitely many nodes, each with a finite number of children. Now t_0 is the root node and by hypothesis has an infinite number of descendents but only a finite number of children. If every one of these children had a finite number of descendents, t_0 would have a finite number of descendents as well. Thus, at least one of its children must have an infinite number of descendents. Choose t_1 to be any one of those children.

Consider any node t_n with an infinite number of descendents. By similar reasoning, at least one of its finitely many children must have an infinite number of descendents. Choose any one of these children to be t_{n+1}. Thus, by induction, there is an infinite set of nodes $\{t_0, t_1, t_2, \cdots\}$ such that t_0 is the root node, t_{n+1} is a child of t_n, and each t_n has an infinite number of descendents. This set forms an infinite branch Γ of T. ▮

Hence the contrapositive of the König Tree Theorem states that if there is no infinite branch of a tableau T, then either T has a finite number of nodes or a node has an infinite number of children. If every branch of T is contradictory (and thus finite) the contrapositive says that if every branch is finite then T has a finite number of nodes.

Theorem 5.12 *If T is a tableau whose every branch is either finished or contradictory, then either T has at least one finished branch or T is a tableau confutation.*

Proof: Suppose T has no finished branches. Then every branch must be contradictory and finite. By the contrapositive of the König Tree Theorem, as there are no infinite branches and every branch is contradictory and finite, there must be a finite number of nodes and thus T must be a finite tableau. Thus, as every branch is contraditory and T is finite, by definition 4.23, T is a tableau confutation.

If T has a least one finished branch then it cannot be contradictory and by definition 4.23, T is a not a tableau confutation. ▮

Theorem 5.13 *Let H be a finite or countably infinite set of sentential logic or first order logic wffs. Then either H has a tableau confutation (in which no free variables occur in first order logic) or H has a model.*

Proof: Let$H$ be a finite or countably infinite set of wffs and suppose H does not have a tableau confutation. By theorems 5.8, 5.9, and 5.10, H does have a tableau T (in which no free variables occur in first order logic) whose every branch is either finished or contradictory and finite. Since, by assumption, T is not a confutation, theorem 5.12 tells us T must have at least one finished branch. Thus by theorem 4.12, the finished branch, and thus the hypothesis set H, must have a model.

If H does not have a model, then T cannot have a finished branch and thus by theorems 5.8, 4.12, and 5.10, every branch must instead be contradictory and finite. ▮

The Extended Completeness Theorem is a direct result.

Theorem 5.14 The Extended Completeness Theorem: *If a wff α is a semantic consequence of a finite or countably infinite set of wffs H, then there is a tableau proof of α from H, i.e.* $H \models \alpha \Rightarrow H \vdash \alpha$.

Proof: Let α be a semantic consequence of the finite or countably infinite set of wffs H. Then $H \cup \{\neg\alpha\}$ has no models. Then by theorem 5.13, $H \cup \{\neg\alpha\}$ must have a tableau confutation, that is, a tableau proof of α from H. ∎

When $H = \emptyset$, then $\emptyset \models \alpha$ implies α is valid and the theorem becomes the Completeness Theorem given at the beginning of this section.

5.3 Examples

Consider soundness and completeness in the axiomatic systems of Zermelo-Fraenkel set thoery, group theory, and Peano Arithmetic. Recall that a theorem in an axiomatic system is a true wff whose validity has been shown through a tableau proof whose hypothesis set H is the axioms of the system. In other words, if H gives a tableau proof of α ($H \vdash \alpha$), then α is a semantic consequence, that is, true in the same models, of H ($H \models \alpha$). Thus axiomatic systems are sound by the Extended Soundness Thoerems. Note this is also a matter of semantics; the axioms of each system are assumed to be true and the thoerems are provably true by that assumption.

Is it possible an axiomatic system may prove both a wff α and its negation $\neg\alpha$? Such an occurence would be disastrous, as a system which can prove a wff and its negation can prove anything at all. Thus it is desirable to prove (from within) that a given axiomatic system is consistent.

Definition 5.15 A axiomatic system is **consistent** if there is no wff α such that α and $\neg\alpha$ are provable from its axioms. A axiomatic system is **$\boldsymbol{\omega}$-consistent** if there is no wff $\alpha(x)$, where x is a free variable, such that $\alpha(x)$ is true for each x and $\exists x(\neg\alpha(x))$ is true.

An axiomatic system T which is ω-consistent is also consistent. An ω-consistent system not only avoids proving a contradiction but also avoids proving both an infinite sequence of wffs (by induction, usually) and the existence of a contradiction of one of those wffs. Kurt Gödel introduced ω-consistency in his proof of the Incompleteness Theorems, the subject of the next chapter. His proof also showed Peano Arithmetic, indeed any axiomatic systems sufficiently powerful enough to be expressed by natural numbers (that is, by a Gödel numbering as explained in the next section), cannot be sound, consistent, and complete. Thus the task of proving consistency and completeness of such systems will not be attempted here, but rather addressed at the end of the following chapter.

Chapter 6

Gödel's Incompleteness Theorems

This chapter is concerned with axiomatic systems which are sound, ω-consistent, complex enough to perform arithmetic, and whose axioms can be listed using an algorithm (such as a computer program). Gödel's First Incompleteness Theorem establishes the existence of a wff $\mathcal{G}$ in such a system T which is true but not provable in the system, implying T cannot be complete. This is done in three steps: expressing every wff of T as a natural number, creating a wff which asserts its own unprovability, and proving such a wff is true metamathematically. Gödel's Second Incompleteness Theorem is a corallory of the first and establishes that if T asserts its own consistency then T must be inconsistent.

For the remainder of this paper, let T denote a sound, ω-consistent axiomatic system able to express elementary arithmetic ("sufficiently complex") and whose axioms can be listed using an algorithm ("effectively generated").

6.1 Gödel Numbering

Gödel numbering is a method of describing wffs (and strings) of T as natural numbers. Each individual symbol of T is assigned a natural number. For example, various symbols of Peano Arithmetic can be labelled as follows:

0	x	$+$	$=$	$($	$)$	s	$\cdot$	$\neg$	$\vee$	$\forall$
1	2	3	4	5	6	7	8	9	10	11

Thus every string of length n can be expressed as a sequence of natural numbers $q_1, q_2, \cdots, q_n$. The Gödel number of such a wff is given by the product $p_1^{q_1} \cdot p_2^{q_2} \cdots p_n^{q_n}$ where p_i is the ith prime number ($p_1 = 2, p_2 = 3, p_3 = 5, etc$). For example, the wff $0 + x = x$ corresponds to the Gödel number $2^1 \cdot 3^3 \cdot 5^2 \cdot 7^4 \cdot 11^2$ $= 392203350$. Note that this method gives every wff a unique Gödel number. By the Fundamental Theorem of Arithmetic, also known as the Unique-Prime-Factorization Thoerem, every natural number can be factored into the form $p_1^{q_1} \cdot p_2^{q_2} \cdots p_n^{q_n}$. Thus every Gödel number can be deconstructed back into a wff of T. For instance, the number 1732825710 can be factored to $2^1 \cdot 3^8 \cdot 5^1 \cdot 7^4 \cdot 11^1$, which can be deconstructed to $0 \cdot 0 = 0$ by the given symbol assignments in PA.

Although every Gödel number is unique, the Gödel numbering is not as there are infinitely many possible assignments. Gödel had a different numbering system where he associated each "variable of type n" (n-ary predicate) with a number of the form p^n where p is a prime strictly greater than 13. This type of system is based on prime factorization, but there are other methods as well, as illustrated by examples 6.1 and 6.2.

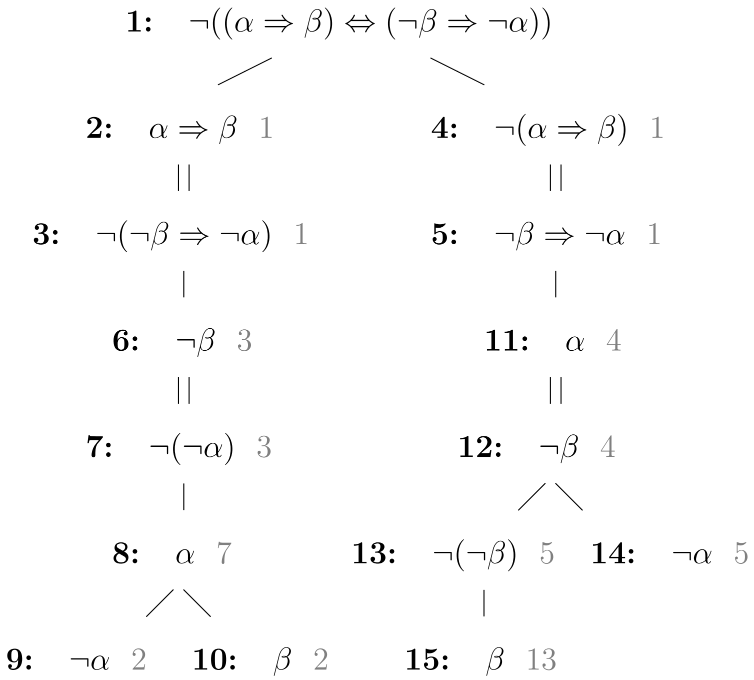
Example 6.1 H. Jerome Keisler in *Mathematical Logic and Computablity* also assigned natural numbers to each symbol, but the natural number which represented each string was deconstructed in an entirely different way. Rather than prime factorization, each decimal place gave an indication of which symbols the string contained (where the places are counted 4321, and 4 is in the fourth decimal place). The even decimal positions marked the beginning and endings of various symbols, where the number 1 indicated a continuation of a symbol and a 2 indicated the beginning of a new symbol. The odd positions contained the numbers corresponding to the various symbols. For example, with the symbol labelling given previously, in Keisler's notation, $\forall x(x = x)$ becomes

$$\large 1 \, {\small 1} \, 1 \, {\small 2} \, 2 \, {\small 2} \, 5 \, {\small 2} \, 2 \, {\small 2} \, 4 \, {\small 2} \, 2 \, {\small 2} \, 6$$

(the actual numbers corresponding to the symbols are enlarged). To ensure every natural number is a Gödel number, the single digits 0-9 are considered to be the empty sequence, any number greater than 2 in an even decimal place is treated as 2, and the number 0 in an even decimal place acts as as 1. [12, pp. 208-210]

Example 6.2 Douglas Hofstadter in his novel *Gödel, Escher, Bach*, assigned each symbol a three digit natural number justified by his own, rather interesting, reasoning. For example, he gave 0 the natural number 666 as it is the number of the beast and = and ∃ the numbers 111 and 333, respectively, because they 'looked' like the symbols. [10, pg.68]

Finite sequences of strings or wffs, such as those which comprise a tableau proof, can also be represented by a (exremely large) Gödel number as well. For instance, consider the tableau proof of the equivalence between an implication and its contrapositive, given in example 4.24 and listed again here for convenience.



It is possilbe to express this proof as a sequence of Gödel numbers, each corresponding to the wff of a numbered node. For example, label the symbols of the tableau in the following way

α	β	$\neg$	$\Rightarrow$	$\Leftrightarrow$	(	)	1	2	$\cdots$	15
1	2	3	4	5	6	7	8	9		22

Then $(\alpha \Rightarrow \beta) \Leftrightarrow (\neg\beta \Rightarrow \neg\alpha)$ has the Gödel number:

$$2^6 3^1 5^4 7^2 11^7 13^5 17^2 19^7 23^5 29^6 31^3 37^2 41^4 43^3 47^1 53^7$$

Let each wff of the tableau proof be preceded by its associated node number and denoted by the corresponding Gödel number, for example, $1\neg((\alpha \Rightarrow \beta) \Leftrightarrow (\neg\beta \Rightarrow \neg\alpha))$ has the Gödel number:

$$2^8 3^3 5^6 7^6 11^1 13^4 17^2 19^7 23^5 29^6 31^3 37^2 41^4 43^3 47^1 53^7 59^7$$

Each such wff and node of the tableau can be listed as one string:

$$1\neg((\alpha \Rightarrow \beta) \Leftrightarrow (\neg\beta \Rightarrow \neg\alpha))2\alpha \Rightarrow \beta 3 \cdots 15\beta$$

This string can be expressed by the following Gödel number:

$$2^8 3^3 5^6 7^6 11^1 13^4 17^2 19^7 23^5 29^6 31^3 37^2 41^4 43^3 47^1 53^7 59^7 61^9 67^1 71^4 73^2 79^{10} \cdots 373^{22} 379^2$$

As the wffs themselves do not contain natural numbers, the numbers corresponding to the nodes mark the begining of a new wff from inside the string. Thus the above Gödel number corresponds to a proof for the equivalence of an implication and its contrapositive.

Gödel numbering allows wffs to reference proofs, other wffs, and, indirectly (for now), themselves via their Gödel numbers. For instance, consider a wff $\alpha(x)$ of T with one free variable whose Gödel number is a. Then the substitution (x//a) produces a wff $\alpha(a)$ which indirectly refers to its (previous) self. Note that $\alpha(a)$ is a different wff than $\alpha(x)$ and thus has a different Gödel number. However, is it possible for a wff to directly reference itself? That is, does a wff $\alpha(a)$ whose Gödel number is also a exist? This question arises, and is answered, in the next section when the Gödel sentence is constructed.

6.2 The Gödel Sentence

From the previous section it is known natural numbers can represent both proofs and wffs. If x is the Gödel number of a sequence of strings and y is the Gödel number of a wff, then a binary predicate $PRF(x, y)$ can be defined whose associated n-ary relation expresses the statement x proves y (in T), that is $x \vdash y$. Then $PRF(x, y)$ is true when x does prove y and false when it does not, i.e. when $(x, y) \in PRF^T(x, y)$. This predicate corresponds to a numerical relationship between x and y. In the previous section it was observed

$$2^8 3^3 5^6 7^6 11^1 13^4 17^2 19^7 23^5 29^6 31^3 37^2 41^4 43^3 47^1 53^7 59^7 61^9 67^1 71^4 73^2 79^{10} \cdots 373^{22} 379^2$$

was the Gödel number of a tableau proof of a wff with the Gödel number

$$2^6 3^1 5^4 7^2 11^7 13^5 17^2 19^7 23^5 29^6 31^3 37^2 41^4 43^3 47^1 53^7$$

By observing the powers in both numbers, it is evident the wff is embedded in the beginning of the proof (indeed, it occurs after the symbols for 1, $\neg$, and (have occurred). $PRF(x, y)$ corresponds to this numerical relation between x and y.

Definition 6.3 A **proof pair** (x, y) is an ordering of any two Gödel numbers x and y such that $PRF(x, y)$ is true.

Consider a wff $\mathcal{G}$ with a Gödel number g. Then it is possible to contruct another wff $\neg\exists x PRF(x, g)$ which states there is no sequence of strings that proves $\mathcal{G}$. Now, suppose $\mathcal{G}$ is the wff $\neg\exists x PRF(x, g)$, i.e. $\neg\exists x PRF(x, g)$ has the Gödel number g. Then $\mathcal{G}$ is a wff which asserts its own unprovability. Such as wff is called a Gödel sentence, which is why it is denoted by $\mathcal{G}$. Of course, finding this natural number g is difficult as it occurs in the wff and thus will affect any attempt at calculating it. Thus it is necessary to prove g can be found, and exists.

Let $\alpha_i(x)$ be any wff which includes x as a free variable. Now every string, wff, and collection of wffs (e.g. proofs) of T can be represented by a unique Gödel number. As Gödel numbers are natural numbers and natural numbers are countable, the number of wffs from T are also countable. Thus every $\alpha_i(x)$'s can be counted and numbered and the $\alpha_i(x)$ for every $x \in \mathbb{N}$ can be viewed through Cantor's diagonal argument in the table below.

	$x=1$	$x=2$	$x=3$	$\cdots$	$x=n$	$\cdots$
$i=1$	$\alpha_1(1)$	$\alpha_1(2)$	$\alpha_1(3)$	$\cdots$	$\alpha_1(n)$	$\cdots$
$i=2$	$\alpha_2(1)$	$\alpha_2(2)$	$\alpha_2(3)$	$\cdots$	$\alpha_2(n)$	$\cdots$
$i=3$	$\alpha_3(1)$	$\alpha_3(2)$	$\alpha_3(3)$	$\cdots$	$\alpha_3(n)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$i=n$	$\alpha_n(1)$	$\alpha_n(2)$	$\alpha_n(3)$	$\cdots$	$\alpha_n(n)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Thus this table gives every wff with a single free variable x. Each wff also has a Gödel number which can be denoted by $a_i(x)$ for $\alpha_i(x)$. Consider the wff $\neg\exists y PRF(y, a_x(x))$ which says that $\alpha_x(x)$ (the wffs on the diagonal) cannot be proven. However, this is a wff with a single free variable x and thus must be $\alpha_n(x)$ for some n and listed in the table. Thus it is able to describe itself as unprovable, as when $x = n$, $\alpha_n(n)$ is the wff $\neg\exists y PRF(y, a_n(n))$. $\mathcal{G}$ is thus $\alpha_n(n)$ and the Gödel number g is $a_n(n)$. Hence $\mathcal{G}$ exists. However, is $\mathcal{G}$ true or false? This is proven by Gödel's First Incompleteness Theorem.

6.3 Gödel's Incompleteness Theorems

Gödel First Incompleteness Theorem was proposition VI from his *On Formally Undecidable Propositions of Principia Mathematica and Related Systems.* Now that $\mathcal{G}$ has been constructed and shown to exist, the actual proof of Gödel's First Incompleteness Theorems is fairly short and simple.

Theorem 6.4 Gödel's First Incompleteness Theorem: *Every sound and ω-consistent axiomatic system complex enough to support Gödel numbering contains a statement which is true, but not provable in the system.*

This theorem simply states that our previously defined axiomatic system T is not complete.

Proof: Let T be an axiomatic system and $\mathcal{G}$ a wff in T as previously defined. If $\mathcal{G}$ is false, then $\neg\big(\neg\exists y PRF(y,g)\big)$, or $\exists y PRF(y,g)$, is true and $\mathcal{G}$ must be provable. As T is sound, every wff with a proof is true, thus $\mathcal{G}$ must be true and hence unprovable, a contradiction in T as it is consistent. Thus $\mathcal{G}$ cannot be false and therefore must be true, but unprovable. Therefore there exists a statement $\mathcal{G}$ in T which is true but not provable. ∎

Thus Gödel's First Incompleteness Theorem is proven. Note that $\mathcal{G}$ is true by metamathematical reasoning; no proof exists within the system T of $\neg\exists y PRF(y,g)$.

Theorem 6.5 Gödel's Second Incompleteness Theorem: *Any sound, sufficiently complex, and effectively generated axiomatic systemt can prove its own consistency if and only if it is inconsistent.*

Gödel's Second Incompleteness Theorem states that our previously defined axiomatic system T (where T does not have an assumption of consistency) can only prove its consistency if it is inconsistent.

Proof: Gödel's Second Incompleteness Theorem is an easy corollary of the first. Let $\mathcal{G}$ remain as defined and T be as previously defined except without ω-consistentcy. Let α be the proposition *T is consistent.* If α is true then the wff $\mathcal{G}$ exists in T. Recall $\vdash \alpha$ indicates there is a proof for α. Then if α is true:

1: $\alpha \Rightarrow \mathcal{G}$ is true by Gödel's First Incompleteness Theorem.
2: $\vdash \alpha \Rightarrow \vdash \mathcal{G}$ by 1 and implication.
3: $\nvdash \mathcal{G} \Rightarrow \nvdash \alpha$ by contrapositive of 2.
4: $\nvdash \mathcal{G}$ is true by Gödel's First Incompleteness Theorem.
5: $\nvdash \alpha$ by modus ponens applied to 3 and 4.

Therefore a proof of the consistency of T does not exist in T. Verbally, this argument goes as follows where T is assumed to be consistent.

1: By Gödel's First Incompleteness Theorem, a consistent system T contains $\mathcal{G}$.
2: Thus if there is a proof of T's consistency then there is a proof for $\mathcal{G}$. This follows from, given two wffs α and β, $(\alpha \Rightarrow \beta) \Rightarrow (\vdash \alpha \Rightarrow \vdash \beta)$.
3: The contrapositive of this implies if $\mathcal{G}$ is unprovable then T's consistency is unprovable.
4: $\mathcal{G}$ is unprovable by Gödel's First Incompleteness Theorem.
5: Thus, by modus ponens, the consistency of T cannot be proven.

Thus T can only prove its consistency if it is inconsistent. ∎

Note that it is only impossible to prove the consistency of T from within. The consistency of a system may have a proof in another, separate system. Of course, then it is necessary to prove that system's consistency.

Together, Gödel's Incompleteness Theorems state that every sound, sufficiently complex, and effectively generated axiomatic system is either incomplete or inconsistent. This means Zermelo-Fraenkel set thoery, group theory, and Peano Arithmetic are either incomplete or inconsistent; either they contain wffs which are true but unprovable or they can prove both a wff and its negation.

Chapter 7

Conclusion

Prove that the axioms of arithmetic are consistent. Gödel's Incompleteness Theorems establish that such a sound, sufficiently complex, and effectively generated system, if it is complete, can not prove its own consistency. However, there is some debate whether these theorems truly solve Hilbert's second question. Although Kurt Gödel eliminated the possibility of a proof for the axioms of arithmetic from the axioms themselves, a proof may still exist from some other method or system. The Gödel sentence, $\mathcal{G}$, itself is proven in such a way, using reasoning from outside the system of which it is a sentence. However, the reasoning was proven to be both sound and complete. Any new proof for the consistency of arithmetic must meet the same criteria.

Inconsistency or incompleteness; Gödel's Incompleteness Theorems establish the inherent hindrance of axiomatic systems which use arithmetic, raising concerns in mathematics, computer science, and even philosophy. If an axiomatic system is inconsistent, then anything and everything may be proven. If the system is incomplete, mathematicians must keep in mind there may exist unsolvable problems. Effectively generated axiomatic system can be expressed via an algorithm and thus programmed into a computer or Turing machine. In computer science, Gödel's Incompleteness Theorems imply certain internal limitations of computers. Humans, who can work 'outside' a system, will know what the computer doesn't: that certain unprovable wffs are theorems. Philosophically, the question becomes which axioms should be accepted and why? Does the existence of a true yet unprovable sentence justify the use of intuition in mathematics? Certain axiomatic systems, such as those proposed by Tarski for elementary Euclidean geometry have been proven to be complete. Is there any other way to consider arithmetic which avoids incompleteness?

Despite the uncertainty raised by Gödel's Incompleteness Theorems, mathematicians are, for the most part, unaffected. Mathematicians work on in much the same way as before Gödel's Incompleteness Theorems, even knowing someone may be laboring to prove the inconsistency of the system in which they endeavor. Perhaps if the inconsistency of a system was established, mathematicians would use it nevertheless.

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