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The hyperboloid model of hyperbolic geometry

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The Hyperboloid Model of Hyperbolic Geometry

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Abstract

The main goal of this thesis is to introduce and develop the hyperboloid model of Hyperbolic Geometry. In order to do that, some time is spent on Neutral Geometry as well as Euclidean Geometry; these are used to build several models of Hyperbolic Geometry. At this point the hyperboloid model is introduced, related to the other models visited, and developed using some concepts from physics as aids. After the development of the hyperboloid model, Fuchsian groups are briefly discussed and the more familiar models of Hyperbolic Geometry are further investigated.

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Chapter 1

History of Geometry 1

The origins of geometry date back to about 4000 years ago; specifically concerning the Egyptians, geometry consisted of a set of principles, arrived at through experimentation and observation, which they used in construction, astronomy, etc. From Egypt, Thales (640-546 B.C.) brought geometry to Greece, where these principles were generalized, expanded on, and explained; Thales himself took the idea of positions and straight edges in the Egyptian geometry to the more abstract points and lines. Much progress was made in the development of geometry as a branch of mathematics by such as Pythagoras, Hippocrates, and around 300B.C. Euclid.

While attempts had been made to describe thoroughly the Grecian geometry, Euclid's was the most significant attempt even though many flaws existed in Euclid's proofs, definitions, and theorems. Euclid built up and developed his geometry using only five postulates. H.S.M. Coxeter [1] gives these postulates as follows:

I. A straight line may be drawn from any one point to any other point.

II. A finite straight line may be produced to any length in a straight line.

III. A circle may be described with any centre at any distance from that centre.

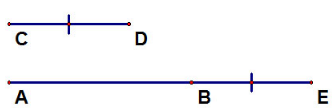
IV. All right angles are equal.

V. If a straight line meet two other straight lines, so as to make the two interior angles on one side of it together less than two right angles, the other straight lines will meet if produced on that side on which the angles are less than two right angles.

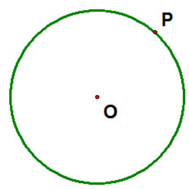
While these may be very near to Euclid's original statement of his five postulates, Greenberg [2] gives a version of the postulates that is perhaps more familiar, and certainly easier to interpret.

I. For every point P and every point Q distinct from P, there exists a unique line l passing through P and Q.

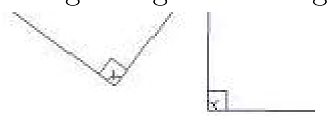
II. For every segment $\overline{AB}$ and every segment $\overline{CD}$, there exists a unique point E such that A, B, and E are collinear with $A - B - E$ and $\overline{BE} \equiv \overline{CD}$.



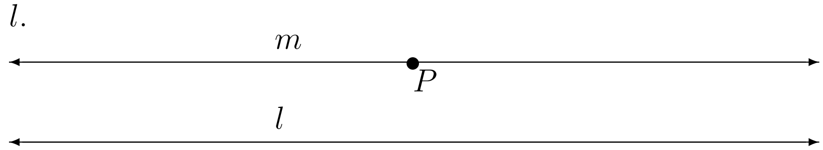
III. For every point O and every point P distinct from O, there exists a unique circle with center O and radius $|\overline{OP}| = d(O, P)$.



IV. All right angles are congruent to each other.



V. For each line l and each point P not on l, there is a unique line m passing through P and parallel to l.



The first four of these were readily accepted by mathematicians when proposed, but the fifth was problematic, and mathematicians attempted to prove this postulate from the first four for centuries. This resulted in the development of Neutral Geometry (a geometry with no parallel postulate), but all attempts failed. Of these, the attempts at a direct proof have been shown to be invalid because they involve circular reasoning; the parallel postulate itself, or an equivalent statement, is assumed at some point during the proof.

Proclus (410-485 A.D.), "one of the main sources of information on Greek geometry" [2], made an early attempt at a proof in which he used circular reasoning by implicitly introducing a new assumption. Coxeter gives Proclus's assumtion as "If a line intersects one of two parallels, it also intersects the other". Other mathematicians such as John Wallis (1616-1703 A.D.) and Alexis Clairaut (171-1765 A.D.) made attempts at a proof by introducing an extra axiom to the first four. Wallis included that given any triangle ΔABC and any segment $\overline{DE}$, there exists a triangle ΔDEF similar to ΔABC ($\Delta ABC \sim \Delta DEF$), while Clairaut assumed that rectangles exist.

Still other mathematicians attempted to prove the Parallel Postulate by contradiction. Included here are Ibn al-Haytham and Girolamo Saccheri, and though these men did not succeed in their attempts to prove Euclid V, many of their results are now theorems in Hyperbolic and Elliptic Geometry.

Saccheri (1667-1733 A.D.) based his work on quadrilaterals $\square ABCD$ in which $d(A, D) = d(B, C)$ with $m\angle DAB = m\angle ABC = \frac{\pi}{2}$. Since $\angle BCD \cong \angle CDA$, Saccheri considered the following three possibilities:

1. The angles $\angle BCD \cong \angle CDA$ are acute.
2. The angles $\angle BCD \cong \angle CDA$ are right angles.
3. The angles $\angle BCD \cong \angle CDA$ are obtuse.

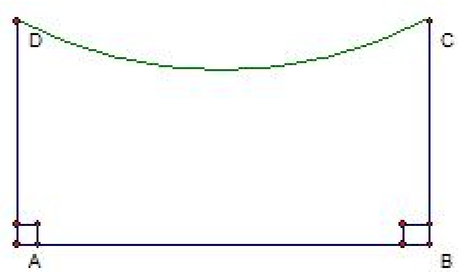


Figure 1.1: Saccheri's acute hypothesis.

Assuming that the first and third of these would lead to contradictions, Saccheri believed he could prove the parallel postulate. While he discovered that the obtuse hypothesis led to a contradiction[2], the acute hypothesis, on the other hand, lead him to deductions important to what would later be called Hyperbolic Geometry, but never to a contradiction [1].

Johann Lambert (1728-1777 A.D.) took a similar course, considering one half of Saccheri's quadrilaterals (halved by the midpoints of sides AB and CD), which was the same course taken by al-Haytham. Lambert came to the same conclusion as Saccheri regarding the obtuse hypothesis (an equivalent hypothesis of Saccheri's for his own quadrilateral), but took the case of the acute hypothesis even further. In fact, after defining the defect of a polygon Lambert concluded that the defect of a polygon is proportional to it's area, and suggested that the acute hypothesis holds in the case of a sphere of imaginary radius [1]. Another interesting discovery Lambert made while considering the acute hypothesis is that, while only angles, and not lengths, are absolute in Euclidean geometry, when Euclid V is denied lengths are also absolute.

While Lambert and Saccheri both discovered Hyperbolic Geometry, neither of them acknowledged the fact: in fact, Greenberg quotes Saccheri as saying "The hypothesis of the acute angle is absolutely false, because [it is] repugnant to the nature of the straight line!". It was not until about 1813 that C.F. Gauss, and almost 10 years later Janos Bolyai, accepted that Euclid V need not be true. Gauss had spent over 30 years (1792-1829) following the denial of the parallel postulate before he was content that the non-euclidean geometry this led to would be consistent [1]. Bolyai, on the other hand, spent significantly less time before he was convinced and reported to his father Farkas Bolyai, in 1823, "out of nothing I have created a strange new universe"[2].

While Gauss had never published his investigation into this non-euclidean geometry, Janos Bolyai published his discoveries in an appendix to his father's book, the *Tentamen*, in 1831. However, another mathematician, Nikolai Lobachevsky, was the first to publish any work on what he called "imaginary" geometry (and later "pangeometry") in 1829. Lobachevsky's first published work received little attention outside of Russia, and in Russia was rejected.[1] Nonetheless, Lobachevsky continued to publish his findings and challenge the Kantian view that Euclidean Geometry is a necessary truth, "inherent in the structure of our mind"[2].

In their investigation into Euclid V and non-euclidean geometry, these men had considered denials of the parallel postulate, of which there are two.

First, consider the Euclidean Parallel Postulate (equivalent to Euclid V, as we will see later):

> For each line l and each point P not on l, there exists a unique line m passing through P and parallel to l.

Then for the proper denials, we have:

1. There exists a line l and a point P not on l such that no line m passing through P is parallel to l.

2. There exists a line l and a point P not on l such that two distinct lines m_1 and m_2 exist which pass through P and are parallel to l.

In considering these denials of Euclid V Janos Bolyai, Lobachevsky, and Gauss found a new, consistent structure for geometry. The first of these denials was rejected as it violated the assumed postulates , while the second led to Hyperbolic Geometry[2]. We will investigate and better define this Hyperbolic Geometry further, but first we'll build up a little Neutral, or Absolute, Geometry.

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[1] Greenberg quotes a Russian journal calling Lobachevsky's discoveries "false new inventions".[2]

[2] Hyperbolic Geometry is also known as Lobachevskian Geometry, likely due to Lobachevsky's investigations (which went much further than Gauss') and subsequent publications; Bolyai never again published his findings after Gauss responded to Bolyai's appendix by saying that he, Gauss, had already come to the same conclusions and "dare not praise such a work".

Chapter 2

Neutral Geometry and Euclid V

2.1 Axioms and Definitions of Neutral Geometry

Now, we've mentioned Saccheri's approach in attempting to prove Euclid V, and how it involves Neutral Geometry. In this chapter we'll look into this approach, which means we'll need to investigate Neutral Geometry. Neutral Geometry is often introduced with Euclid's first four postulates, but to get things going we'll use the revised version of David Hilbert's axioms as suggested by Wallace and West[1] [3], along with a few terms and relations, which we'll leave undefined:

AXIOM 1: Given any two distinct points, there is exactly one line that contains them.

Notice that Euclid I is given by our first axiom. The rest of our axioms may seems less familiar, though Euclidean Geometry may be built up using them.

[1]This approach is not the most sophisticated, and in fact detracts from the elegance of an axiomatic geometry; some of the following axioms may be derived from the others (this set of axioms is not independent).

AXIOM 2: To every pair of distinct points, there corresponds a unique positive number. This number is called the distance between the two points.

AXIOM 3: The points of a line can be coordinatized with the real numbers such that

1. To every point of the line there corresponds exactly one real number,
2. To every real number there corresponds exactly one point of the line, and
3. The distance between two distinct points is the absolute value of the difference of the corresponding real numbers.

AXIOM 4: Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

The next four axioms deal with space, and though much of what we'll consider will be two dimensional geometry, they are important to include for completeness.

AXIOM 5: Every plane contains at least three noncollinear points, and space contains at least four nonplanar points.

AXIOM 6: If two points lie in a plane, then the line containing these points lies in the same plane.

AXIOM 7: Any three points lie in at least one plane, and any three noncollinear points lie in exactly one plane.

AXIOM 8: If two (distinct) planes intersect, then that intersection is a line.

AXIOM 9: Given a line and a plane containing it, the points of the plane that do not lie on the line form two sets such that

1. each of the sets is convex and

2. if P is in one set and Q is in the other, then segment $\overline{PQ}$ intersects the line.

The next axiom is an extension of the last and, again, as it deals with space we will not be using it immediately as we consider two-dimensional geometries.

AXIOM 10: The points of space that do not lie in a given plane form two sets such that

1. Each of the sets is convex and
2. If P is in one set and Q is in the other, then segment $\overline{PQ}$ intersects the plane.

The next few axioms are focused on angles, which is probably a familiar concept...

Definition 2.1 Two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (or segments $\overline{AB}$ and $\overline{AC}$) form an *angle* if A, B, and C are not collinear. Such an angle is denoted $\angle BAC$ or $\angle CAB$, and the *measure* of this angle is the amount of rotation about the point A required to make one ray (or segment) coincide with the second.

Definition 2.2 (i) If the sum of the measure of two angles A and B is $180°$, then angles A and B are *supplementary*, while (ii) if the sum of the measure of two angles is $90°$ then the two angles are *complementary*.

AXIOM 11: To every angle there corresponds a real number between $0°$ and $180°$.

AXIOM 12: Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every r between $0°$ and $180°$, there is exactly one ray $\overrightarrow{AP}$ with P in H such that $m\angle PAB = r$.

Definition 2.3 Consider two lines, $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$. We know that each line separates the plane into two convex sets. Let M be the convex set, formed by the separation of the plane by $\overleftrightarrow{AB}$, which contains C. Similarly, let N be the convex set containing B

formed by the separation of the plane by $\overleftrightarrow{AC}$. Then the *interior of the angle* $\angle BAC$ is defined to be $N \cap M$. Similarly, letting L be the convex set containing A which is formed by the separation of the plane by $\overleftrightarrow{BC}$, we define the *interior of* ΔABC to be $L \cap M \cap N$.

AXIOM 13: If D is a point in the interior of $\angle BAC$, then

$$m\angle BAC = m\angle BAD + m\angle DAC.$$

Definition 2.4 Given three distinct points, A, B, and C, B is *between* A and C, $A - B - C$, if $d(A,B) + d(B,C) = d(A,C)$.

Definition 2.5 Given two angles $\angle ABC$ and $\angle CBD$, if $A - B - D$, then these angles form a *linear pair.*

AXIOM 14: If two angles form a linear pair, then they are supplementary.

For the next axiom, a few definitions are necessary:

Definition 2.6 Two segments are *congruent* when their measures are equal.

Definition 2.7 Two angles are *congruent* when their measures are equal.

Definition 2.8 Given two polygons P_1 and P_2, a *one-to-one correspondence* between P_1 and P_2 is a bijection which maps each side of P_1 to a side of P_2. Furthermore, if A and B are sides of P_1 such that A and B meet at a vertex, and X and Y are sides of P_2 such that X and Y meet at a vertex, if A corresponds to X (our bijection maps A to X) and B corresponds to Y, then the included angle of A and B is mapped to the included angle of X and Y

Definition 2.9 Two polygons are *congruent* when corresponding sides and angles are congruent.

AXIOM 15: Given a one-to-one correspondence between two triangles (or between a triangle and itself): If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Notice that we have no axiom regarding parallelism; if we were to include Euclid V we would have Euclidean Geometry. Now, the undefined terms we'll be using are *point*, *line*, and *plane*, while the undefined relation we'll use is "lies on". Though we leave these undefined, a mutual understanding of the meaning of each of these is assumed; as an example of the relation, see the image for the restatement of Euclid V, where the point P *lies on* the line m.

Some more terminology we use is defined in Appendix A.

2.2 Theorems of Neutral Geometry

Now that we know the "rules" of Neutral Geometry, let's explore some of their results.

Theorem 2.10 *The congruence relations defined above are equivalence relations.*

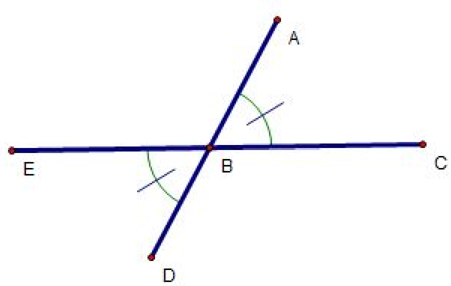
Theorem 2.11 *(i) Every segment has exactly one midpoint, and (ii) every angle has exactly one bisector.*

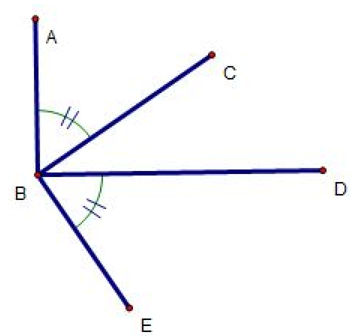
Proof: (i) Consider segment $\overline{AB}$, and using Axiom 4 choose a coordinate system so that the coordinate of A is zero, and B is 2. By Axiom 3 there exists C on line $\overleftrightarrow{AB}$ such that the coordinate of C is 1. Therefore, $d(A, C) = d(C, B) = 1$. Thus, C is the midpoint of segment $\overline{AB}$, and by Axiom 3 part2 this can be the only midpoint.

(ii) Now consider angle $\angle ABC$. Let H be the half-plane containing $\overrightarrow{BA}$ with $\overrightarrow{BC}$ on th edge of H. Given $m(\angle ABC) = \alpha$, by Axiom 12 there exists exactly one ray $\overrightarrow{AD}$ with D in H such that $m(\angle DAB) = \frac{\alpha}{2}$. Since there is exactly one such ray, we have both existence and uniqueness of the angle bisector of arbitrary angle $\angle ABC$. ∎

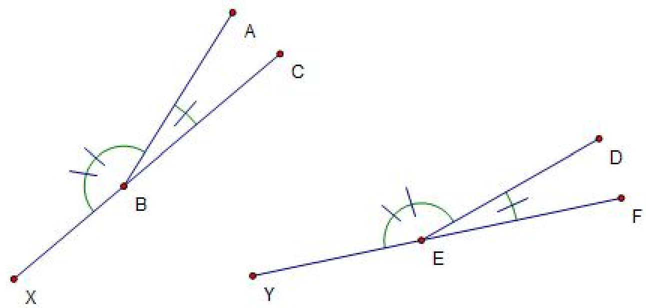
Theorem 2.12 *Supplements and complements of the same or congruent angles are congruent.*

Proof: First let's look at supplementary and complementary angles of a single angle. Suppose angles $\angle ABC$ and $\angle DBE$ are supplementary to angle $\angle CBD$. Then $m(\angle DBE) = 180° - m(\angle CBD) = m(\angle DBE)$. Therefore, $\angle ABC$ is congruent to $\angle DBE$. Similarly, if angles $\angle ABC$ and $\angle DBE$ are complementary to angle $\angle CBD$, then $m(\angle DBE) = 90° - m(\angle CBD) = m(\angle DBE)$ so that $\angle ABC$ is congruent to $\angle DBE$.





Now let's consider two angles, $\angle ABC \cong \angle DEF$. If $\angle XBA$ and $\angle YED$ are supplementary angles of $\angle ABC$ and $\angle DEF$ respectively, then $m(\angle XBA) = 180° - m(\angle ABC) = 180° - m(\angle DEF) = m(\angle YED)$. Therefore, $\angle XBA \cong \angle YED$. Similarly, if $\angle XBA$ and $\angle YED$ are complementary angles of $\angle ABC$ and $\angle DEF$ respectively, then $m(\angle XBA) = 90° - m(\angle ABC) = 90° - m(\angle DEF) = m(\angle YED)$. Therefore, $\angle XBA \cong \angle YED$. ∎



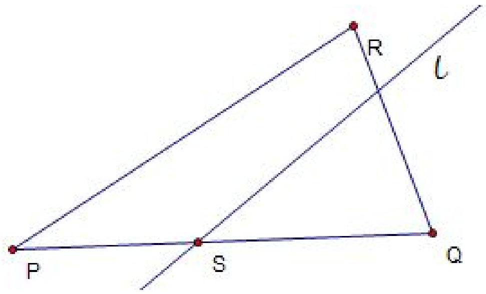
Theorem 2.13 *Given a line l and distinct points P, Q, and R not on l, if P and Q are on the same side of l and if Q and R are on different sides of l, then P and R are on different sides of l.*[2]

Proof: From our axioms, l separates the plane into two convex sets, H and K. Since P and Q are on the same side of l, assume wlog that $P, Q \in H$. Since Q and R are on opposite sides of l, $R \in K$. Since $P \in H$ and $R \in K$, P and R are on different sides of l. ∎

Corollary 2.2.1 *Given a line l and distinct points P, Q, and R not on l, if P and Q are on different sides of l, and Q and R are on different sides of l, then P and R are on the same side of l. If, instead, P and Q are on the same side of l, and Q and R are on the same side of l, then P and R are also on the same side of l.*

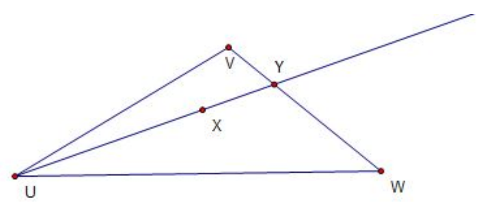
Theorem 2.14 Pasch's Axiom *If a line l intersects ΔPQR at a point S such that $P - S - Q$, where l is distinct from $\overleftrightarrow{PQ}$, then l intersects segments PR or RQ.*

[2] See the footnote to the introductory paragraph of §2.1; given an independent set of axioms, this theorem leads to axiom 9.



Proof: From the hypothesis we know that P and Q are on opposite sides of l since $\overline{PQ}$ intersects l at S and $P - S - Q$. The, wlog, assume that l does not intersect $\overline{PR}$. Then P and R are on the same side of l, which means that R and Q are on opposite sides of l, so that there is T such that$R - T - Q$ and $\overline{RQ}$ intersects l at T. Hence, if l intersects one side of a triangle, and does not intersect one of the remaining sides it must intersect the last. ∎

Theorem 2.15 The Crossbar Theorem. *If X is a point in the interior of ΔUVW, then ray UX intersects segment WV at a point Y such that $W - Y - V$.*



Proof: Let X be in the interior of ΔUVW, and notice that $\overrightarrow{UX}$ cannot intersect $\overline{UV}$ or $\overline{UW}$ at a point Y distinct from U or Axiom 1 would be contradicted; if $\overrightarrow{UX}$ intersects $\overline{UV}$ (or $\overline{UW}$) at some point Y distinct from U, then we have distinct lines $\overleftrightarrow{UV}$ (or $\overleftrightarrow{UW}$) and $\overleftrightarrow{UX}$ both passing through U and Y. So either $\overrightarrow{UX}$ intersects $\overline{VW}$ at a point Y where $W - Y - V$, or it does not exit ΔUVW (it does not intersect any side of the triangle, except at U from which point the ray emanates).

To show that $\overrightarrow{UX}$ must intersect $\overline{VW}$, we will first use Pasch's Axiom to show that $\overleftrightarrow{UX}$ must intersect $\overline{VW}$ and then argue that, for $X' - U - X$, $\overrightarrow{UX'}$ cannot intersect $\overline{VW}$. Take T to be a point of $\overleftrightarrow{UW}$ such that $T-U-W$, and R to be a point of $\overleftrightarrow{UV}$ such that $R-U-V$. By Pasch's Axiom, $\overleftrightarrow{UX}$ must intersect $\overline{TV}$ or $\overline{VW}$, and it must intersect $\overline{RW}$ or $\overline{VW}$.

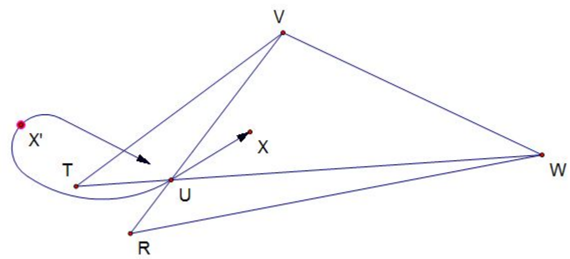


Figure 2.1: Crossbar Theorem Figure 1

From our set-up, we know that T and W are on opposite sides of $\overleftrightarrow{UX}$, and that R and V are on opposite sides of $\overleftrightarrow{UX}$; it follows that either T and V are on the same side of $\overleftrightarrow{UX}$ or V and W are on the same side of $\overleftrightarrow{UX}$. For the sake of contradiction, suppose that V and W are on the same side of $\overleftrightarrow{UX}$, so that $\overleftrightarrow{UX}$ must intersect $\overline{TV}$ at some point Y where $T-Y-V$. However, if T and V are on opposite sides of $\overleftrightarrow{UX}$, then R and W must also be on opposite sides of $\overleftrightarrow{UX}$, so $\overleftrightarrow{UX}$ must also intersect $\overline{RW}$ at some point Z with $R-Z-W$. Since we've confirmed that $\overrightarrow{UX}$ cannot intersect $\overline{UW}$ or $\overline{UV}$, and we've assumed that $\overleftrightarrow{UX}$ (and therefore $\overrightarrow{UX}$) does not intersect $\overline{VW}$, $\overrightarrow{UX}$ must not intersect $\overline{RW}$, $\overline{WV}$, or $\overline{TV}$. Letting X' be a point such that $X'-U-X$, it must then be that $\overrightarrow{UX'}$ intersects $\overline{TV}$ at Y and $\overline{RW}$ at Z. For all of this to occur, axiom 1 must be violated; wlog supposing that $Z-Y-X'-U$, $\overrightarrow{UX'}$ enters ΔTUV at Y, and in order to pass through Z it must exit through $\overline{TV}$, $\overline{UV}$, or $\overline{TU}$, and in each case axiom 1 is violated. Therefore, V and W must be on opposite sides of $\overleftrightarrow{UX}$.

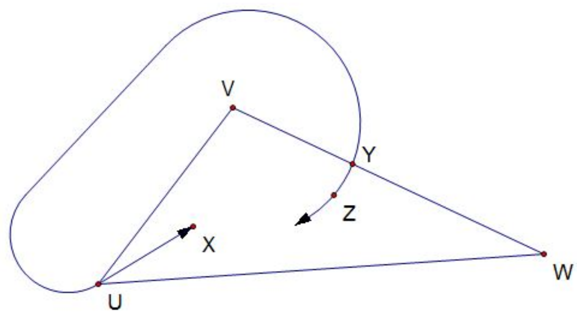
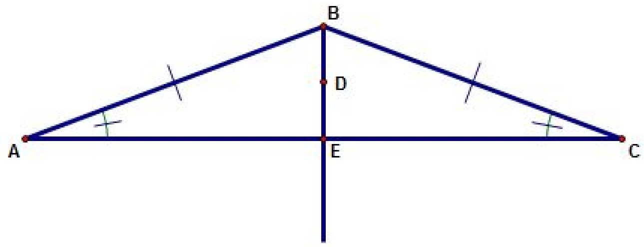


Figure 2.2: Crossbar Theorem Figure 2

> Here we see X and Y, points which are in the interior of the triangle while on different sides of the line passing through $\overleftrightarrow{VW}$, contradicting our definition of the interior of a triangle.

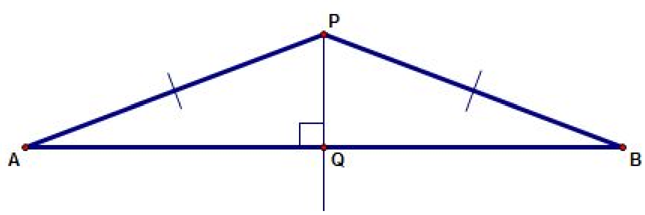
Now, let Y be the point at which $\overleftrightarrow{UX}$ intersects $\overline{VW}$, where $V - Y - W$. Recall that the interior of a triangle is defined the the intersection of three half planes, and for ΔUVW these half planes are determined by the lines of which $\overline{UV}$, $\overline{VW}$, and $\overline{UW}$ are segments. From this it follows that the situation described in figure 2.2 is impossible; it cannot be that $X - U - Y$, or there would exist Z such that $X - U - Y - Z$, where Z is interior to ΔUVW, but $\overline{XZ}$ passes through $\overleftrightarrow{VW}$, so X and Z would be on different sides of $\overleftrightarrow{VW}$. Therefore, $U - X - Y$, so $\overrightarrow{UX}$ intersects $\overline{VW}$ at a point Y between V and W as desired. ∎

Theorem 2.16 The Isosceles Triangle Theorem. *If two sides of a triangle are congruent, then the angles opposite those sides are congruent.*

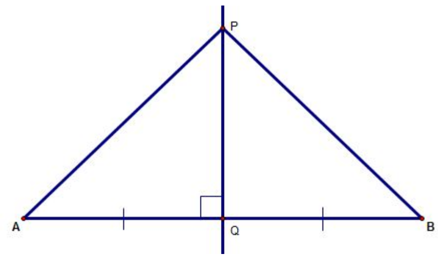


Proof: Suppose sides $\overline{AB}$ and $\overline{BC}$ of triangle ΔABC are congruent. By Theorem 2.11 there exists angle bisector $\overrightarrow{BD}$ of $\angle ABC$, and by the Crossbar Theorem $\overrightarrow{BD}$ must intersect $\overline{BC}$ at some point E. Then by Theorem 2.10, the definition of angle bisector, our hypothesis, and Axiom 15 (the SAS axiom) triangles ΔABE and ΔCBE are congruent. Therefore, $\angle BAC \cong \angle BCA$. ∎

Theorem 2.17 *A point is on the perpendicular bisector of a line segment if and only if it is equidistant from the endpoints of the line segment.*



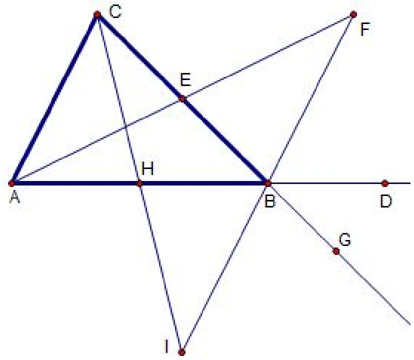
Proof: Given segment $\overline{AB}$ and point P, first suppose that P is equidistant from both A and B. Then, as in the proof of Theorem 2.16, there is Q on $\overline{AB}$ such that the angle bisector of $\angle APB$ intersects $\overline{AB}$ at Q. Then by Theorem 2.10, the definition of angle bisector, our hypothesis, and Axiom 15 $\Delta APQ \cong \Delta BPQ$. Then we have that $\overline{AQ} \cong \overline{BQ}$, and ray $\overrightarrow{PQ}$ bisects $\overline{AB}$. Furthermore, since $\angle AQP \cong \angle BQP$ and these two angles are supplementary, they must be right angles. Therefore, P lies on the perpendicular bisector of $\overline{AB}$.



Now, from the other direction, assume that P lies on the perpendicular bisector of $\overline{AB}$. If P does not lie on $\overline{AB}$, then let Q be the intersection of the perpendicular bisector of $\overline{AB}$ with $\overline{AB}$. Then by Axiom 15, the definition of bisector, and Theorem 2.10, $\Delta AQP \cong \Delta BQP$. Thus, $\overline{AP} \cong \overline{BP}$, and by the definition of segment congruence P is equidistant from both A and B. If P lies on $\overline{AB}$, then by the definition of perpendicular bisector, P is equidistant from A and B. ∎

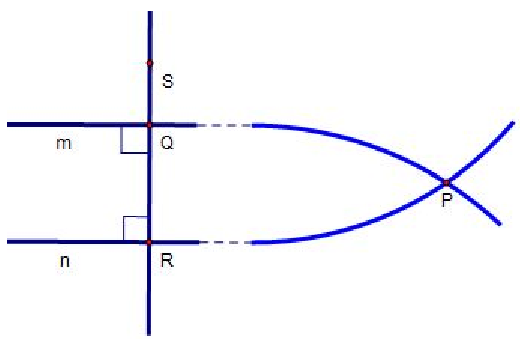
Theorem 2.18 The Exterior Angle Theorem. *Each exterior angle of a triangle is greater in measure than either of the nonadjacent interior angles of the triangle.*

Proof: Consider triangle ΔABC; by Axioms 3 and 4 we may take a point D such that $A - B - D$. Here, $\angle CBD$ is an exterior angle of $\angle ABC$. Now, let E be the midpoint of $\overline{BC}$, and by Axiom 4 we have a point F such that $A - E - F$ and $\overline{AE} \cong \overline{EF}$. Notice that $\angle CEA$ and $\angle FEB$ are supplementary to $\angle CEF$, and so by Theorem 2.12 are congruent. Thus by Axiom 15 we have $\Delta AEC \cong \Delta FEB$. Now, since F is interior to $\angle CBD$, we have that $m\angle ACB = m\angle CBF < m\angle CBD$.



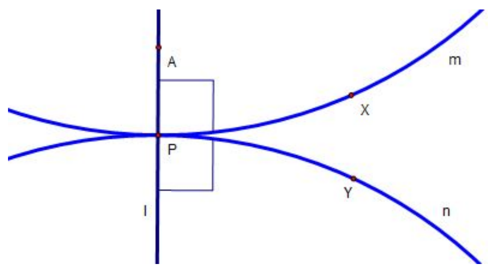
Then by symmetry we have $m\angle CAB < m\angle ABG$. Since $m\angle ABG = m\angle CBD$, we have $m\angle CAB < m\angle CBD$. ∎

Theorem 2.19 *If line l is a common perpendicular to lines m and n, then m and n are parallel.*



Proof: Suppose instead that m and n are not parallel. Then either m and n intersect on l or off l. First, let P be the intersection of m and n, with P not on l. Then m intersects l at a point Q and n intersects l at a point R distinct from Q. This gives us a triangle, ΔPQR, where $\angle PQR \cong \angle PRQ$ are right angles. Letting S be a point on l such that $S - Q - R$, we have that $\angle PQS$, an exterior angle of ΔPQR must also be a right angle. This contradicts Theorem 2.18 since we have $m\angle PQS = m\angle PRQ$. So if m and n are not parallel, then their point of intersection must lie on l.

Now assume that P, the point of intersection of m and n lies on l. Let l be the boundary of a half-plane H, take any point A distinct from P on l, and choose two points X and Y on m and n respectively so that X and Y lie in H. Then $m\angle APX = m\angle APY = 90°$; this contradicts Axiom 12. Therefore, distinct lines m and n cannot intersect if they are both perpendicular to line l, so m and n are parallel. ∎

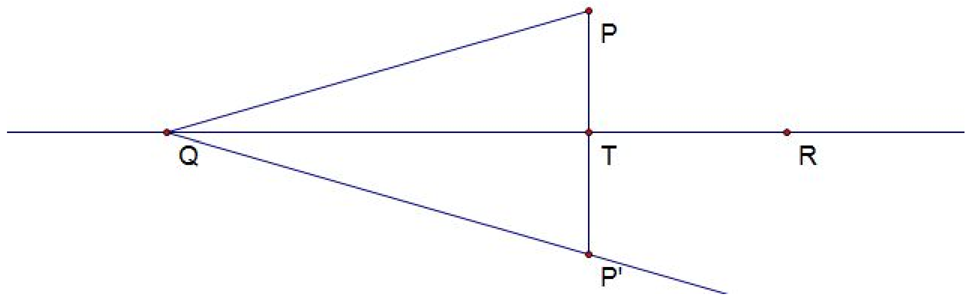


Theorem 2.20 *Given a line l and a point P, there exists one and only one line m through P perpendicular to l.*

Proof: There will be two cases for this: P may lie on l, or P my not lie on l. In either case, Theorem 2.19 guarantees uniqueness; if two lines m and n are perpendicular to l, then they are parallel or they are the same line. If we assume that m and n both pass through P, then m and n must be the same line. We'll consider each case separately for existence.

Now, for the first case Axiom 12 guarantees the existence of a perpendicular; simply choose any point Q on l distinct from P, and Axiom 12 guarantees that there is exactly one ray $\overrightarrow{PR}$ in the half-plane H, whose boundary is l, such that $m\angle QPR = 90°$. Then $\overleftrightarrow{PR}$ is the desired line perpendicular to l passing through P.

For the second case, assume that P does not lie on l and let Q be any point on l. If $\overleftrightarrow{PQ}$ is perpendicular to l, then we're happy. If $\overleftrightarrow{PQ}$ is not perpendicular to l, then take R on l such that $\angle RQP$ is acute. There exists a unique ray $\overrightarrow{QS}$, with S and P on opposite sides of l, such that $\angle RQP \cong \angle RQS$, and there exists P' on $\overrightarrow{QS}$ such that $\overline{QP} \cong \overline{QP'}$. Letting T be the point of intersection of $\overleftrightarrow{PP'}$ with l, by axiom 15 we have $\Delta TQP \cong \Delta TQP'$. Since $\angle PTQ \cong \angle P'TQ$, and $\angle PTQ$ and $\angle P'TQ$ are supplementary angles, they must be right angles. So $\overleftrightarrow{PP'}$ is perpendicular to l. ∎

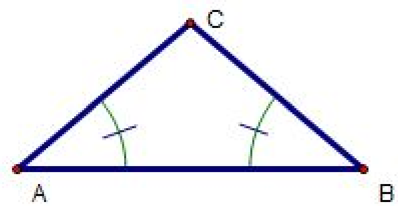


Theorem 2.21 The Angle-Side-Angle Congruence Condition for Triangles. *If two angles and the included side of one triangle are congruent, respectively, to two angles and the included side of a second triangle, then the triangles are congruent.*

Proof: Given triangles ΔABC and ΔDEF such that $\angle ABC \cong \angle DEF$, $\angle BAC \cong \angle EDF$, and $\overline{AB} \cong \overline{DE}$, we have that, by Axiom 15, if $\overline{BC} \cong \overline{EF}$ or $\overline{AC} \cong \overline{DF}$ then $\Delta ABC \cong \Delta DEF$.

Suppose that $\overline{BC} \ncong \overline{EF}$; in particular, let $m\overline{BC} < m\overline{EF}$. Then there is F' on $\overline{EF}$ such that $E - F' - F$ and $\overline{BC} \cong \overline{EF'}$, and by Axiom 15 $\Delta ABC \cong \Delta DEF'$. However, this implies that $\angle EDF' \cong \angle BAC \cong \angle EDF$, which contradicts Axiom 12. Therefore, $\overline{BC} \cong \overline{EF}$, and by Axiom 15 $\Delta ABC \cong \Delta DEF$. ∎

Theorem 2.22 Converse of the Isosceles Triangle Theorem. *If two angles of a triangle are congruent, then the sides opposite those angles are congruent.*

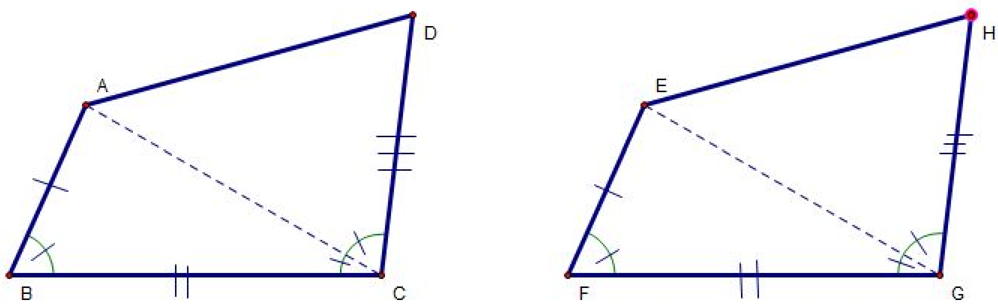


Proof: Given a triangle, ΔABC, such that $\angle CAB \cong \angle CBA$, we have by Theorem 2.21 that $\Delta ABC \cong \Delta BAC$ since any segment is congruent to itself; in particular, $\overline{AB} \cong \overline{BA}$. Therefore, $\overline{AC} \cong \overline{BC}$. ∎

Theorem 2.23 The Angle-Angle-Side Congruence Condition for Triangles. *If two angles and the side opposite one of them in one triangle are congruent to the corresponding parts of the second triangle, then the triangles are congruent.*

Proof: Given two triangles, ΔABC and ΔDEF in which $\angle CAB \cong \angle FDE$, $\angle ABC \cong \angle DEF$, and $\overline{BC} \cong \overline{EF}$., if we were also given that $\overline{AB} \cong \overline{DE}$ then by Axiom 15 the two triangles would be congruent, so assume that $m\overline{AB} < m\overline{DE}$. Then there exists D' on $\overline{DE}$, with $D - D' - E$, such that $\overline{AB} \cong \overline{D'E}$. This implies that $\Delta ABC \cong \Delta D'EF$, and it follows that $\angle FD'E \cong \angle CAB \cong \angle FDE$. This contradicts Axiom 12, so it must be that $\overline{AB} \cong \overline{DE}$, and $\Delta ABC \cong \Delta DEF$. ∎

Theorem 2.24 The Side-Angle-Side-Angle-Side Congruence Condition for Quadrilaterals. *If the vertices of two convex quadrilaterals are in one-to-one correspondence such that the three sides and the two included interior angles of one quadrilateral are congruent to the corresponding parts of a second quadrilateral, then the quadrilaterals are congruent.*



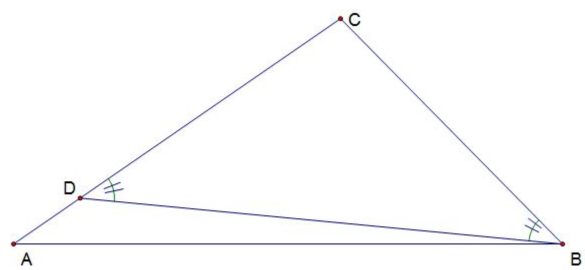
Proof: Given two quadrilaterals, $\square ABCD$ and $\square EFGH$ in which $\overline{AB} \cong \overline{EF}$, $\overline{BC} \cong \overline{FG}$, $\overline{CD} \cong \overline{GH}$, $\angle ABC \cong \angle EFG$, and $\angle BCD \cong \angle FGH$, we will show the congruence of the quadrilaterals as wholes by considering a diagonal of each. First note that, by Axiom 15, $\Delta ABC \cong \Delta EFG$, so that $\overline{AC} \cong \overline{EG}$ and $\angle BCA \cong \angle FGE$. Then

$$m\angle ACD = m\angle BCD - m\angle BCA = m\angle FGH - m\angle FGE = m\angle EGH,$$

so $\angle ACD \cong \angle EGH$ and, by S-A-S congruence, $\Delta ACD \cong \Delta EGH$. This tells us that $\angle GHE \cong \angle CDA$, $\overline{AD} \cong \overline{EG}$, and $\angle CAD \cong \angle GEH$, which then implies that $\angle BAD \cong \angle FEH$. Therefore, $\square ABCD \cong \square EFGH$. ∎

Theorem 2.25 *If two angles of a triangle are not congruent, then the sides opposite them are not congruent, and the larger side is opposite the larger angle.*

Proof: Given a triangle ΔABC in which $\angle CAB \ncong \angle CBA$, by the contrapositive of the converse of the isosceles triangle theorem we know that $\overline{CA} \ncong \overline{CB}$. So either $m\overline{CB} > m\overline{CA}$ or $m\overline{CA} > m\overline{CB}$. Assume that $m\angle CAB > m\angle CBA$, so that our goal is to show that $m\overline{CB} > m\overline{CA}$. Assume instead that $m\overline{CA} > m\overline{CB}$. Then there is D on $\overline{CA}$ such that $A - D - C$ and $\overline{CD} \cong \overline{CB}$. Then $\angle CDB \cong \angle CBD$, and by the exterior angle theorem $m\angle CDB > m\angle DAB$. However, since $m\angle CBA = m\angle CBD + m\angle DBA$, this would imply that $m\angle CBA > m\angle CAB$. Thus, by contradiction it must be that $m\overline{CB} > m\overline{CA}$, so the larger side is opposite the larger angle. ∎



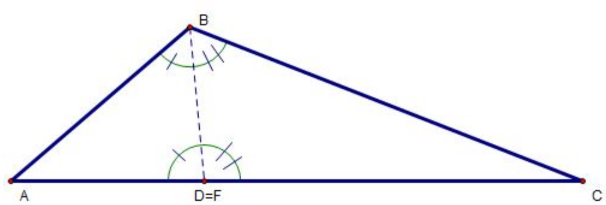
Theorem 2.26 The Inverse of the Isosceles Triangle Theorem. *If two sides of a triangle are not congruent, then the angles opposite those sides are not congruent, and the larger angle is opposite the larger side.*

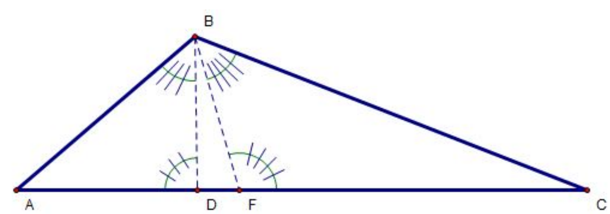
Proof: Suppose that in triangle ΔABC we have that $\overline{AC} \ncong \overline{BC}$. Then the angles opposite these sides are not congruent by the contrapositive of the isosceles triangle theorem. It then follows from the previous theorem that the larger angle is opposite the larger side. ∎

Theorem 2.27 The Triangle Inequality. *The sum of the measures of any two sides of a triangle is greater than the measure of the third side.*

Proof: Given a triangle ΔABC, suppose that $m\overline{AB} + m\overline{BC} \leq m\overline{AC}$. Then there exists D and F, not necessarily distinct, such that $A - D - C$ or, if they are distinct, $A - D - F - C$ such that $\overline{AB} \cong \overline{AD}$ and $\overline{CB} \cong \overline{CF}$.

First, suppose that $D = F$. Then we have that ΔADB and ΔCDB are isosceles triangles such that $\angle ADB \cong \angle ABD$ and $\angle CDB \cong \angle CBD$ by Theorem 2.16. However, since $\angle ADB$ and $\angle CDB$ are supplementary angles, $\angle ABD$ and $\angle CBD$ must also be supplementary angles by congruence. Since $\angle ABD$ and $\angle CBD$ are supplementary and adjacent, they form a linear pair (one ray from each angle, together, form a line). This contradicts Axiom 1 since we would have two distinct lines passing through the same two distinct points A and C.





Now, assuming instead that D and F are distinct, we have that ΔADB and ΔCFB are isosceles triangles in which $\angle ABD \cong \angle ADB$ and $\angle CBF \cong \angle CFB$. If either of $\angle CFB$ or $\angle ADB$ are obtuse, or right, angles, then the exterior angle $\angle BFA$ or $\angle BDC$, respectively, is an acute, or right, angle. This would contradict the exterior angle theorem (2.18), so $\angle ADB$ and $\angle CFB$ are acute angles. Then $\angle BDC$ and $\angle BFA$ are obtuse, and the exterior angle of ΔBDF has measure less than $m(\angle BFD)$, again contradicting Theorem 2.18.

Hence, $m(\overline{AB}) + m(\overline{BC}) > m(\overline{AC})$. ∎

Theorem 2.28 The Hinge Theorem. *If two sides of one triangle are congruent to two sides of a second triangle, and the included angle of the first triangle is larger in measure than the included angle of the second triangle, then the measure of the third side of the first triangle is larger than the measure of the third side of the second triangle.*

Proof: Take triangles ΔABC and ΔDEF, where $\overline{AC} \cong \overline{DF}$, $\overline{AB} \cong \overline{DE}$, and $m(\angle CAB) < m(\angle FDE)$. There is G such that B and G are on opposite sides of $\overleftrightarrow{AC}$, $\angle BAG \cong \angle EDF$, and $\overline{AG} \cong \overline{AC} \cong \overline{DF}$. Then, using that ΔABG is an isosceles triangle,

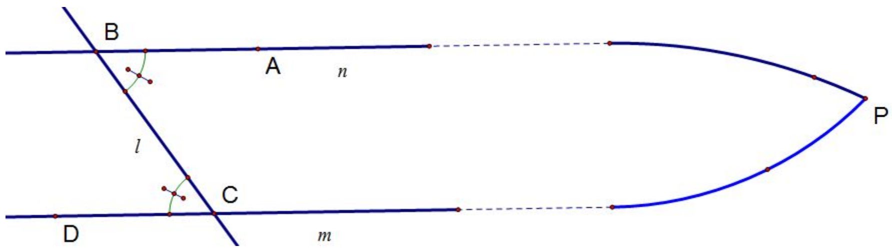
$$m\angle BCG > m\angle ACG = m\angle AGC > m\angle BGC.$$

Then, by Theorem 2.25, and using ΔBCG, $m\angle BCG > m\angle BGC$ implies that $m\overline{BC} < m\overline{BG} = m\overline{FE}$. ∎

Theorem 2.29 The Side-Side-Side Congruence Condition for Triangles. *If all three sides of one triangle are congruent to all three sides of the second triangle, then the triangles are congruent.*

Proof: Given triangles ΔABC and ΔDEF such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{CA} \cong \overline{FD}$, suppose that $\angle ABC \ncong \angle DEF$. Then, by Theorem 2.28 $\overline{AC} \ncong \overline{DF}$. This contradicts our hypothesis, so it must be that $\angle ABC \cong \angle DEF$. Then by Axiom 15 (or by running into a contradiction for a similar assumption for the other two angles of ΔABC), we have that $\Delta ABC \cong \Delta DEF$. ∎

Theorem 2.30 The Alternate Interior Angle Theorem. *If two lines are intersected by a transversal such that a pair of alternate interior angles formed by the intersection are congruent, then the lines are parallel.*



Proof: Given distinct lines m and n, let l be a transversal of m and n such that a pair of alternate interior angles are congruent; in particular, assume that $\angle ABC \cong \angle DCB$, where points A, B, C, and D lie as indicated above. Now, if m and n intersect at some point P, then we have triangle ΔBCP. Notice, however, that if P is on the side of $\overleftrightarrow{BC}$ opposite D, then we have the exterior angle of $\angle BCP$ congruent to angle $\angle CBP$. This violates the Exterior Angle Theorem, and the same contradiction appears if P were on the other side of $\overleftrightarrow{BC}$. Therefore, the intersection P cannot exist, and the lines m and n are parallel. ∎

Theorem 2.31 *The sum of the measure of any two angles of a triangle is less than $180°$.*

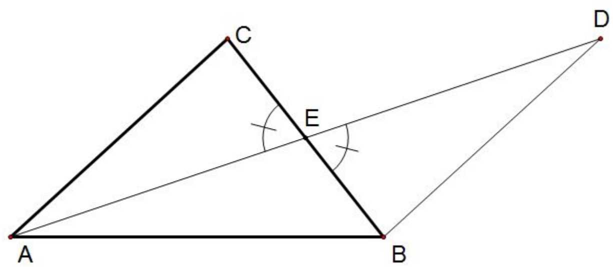
Proof: Given ΔABC, suppose that $m\angle ABC + m\angle CAB \geq 180°$. Then, extending $\overline{AB}$ to D where $A - B - D$, we have that $m\angle CBD = 180° - m\angle ABC \leq m\angle CAB$. This contradicts the exterior angle theorem, so our initial supposition cannot hold. ∎

Theorem 2.32 *For any triangle ΔABC, there exists a triangle ΔAB_0D having the same angle sum as ΔABC, and $m(\angle CAB) \geq 2m(\angle DAB_0)$.*

Proof: Given ΔABC, let E be the midpoint of $\overline{CB}$. There is a point D such that $A - E - D$ and $\overline{AE} \cong \overline{ED}$. By Axiom 15 $\Delta ACE \cong \Delta DBE$. By this congruence, we have the angle sum of ΔABC as

$$m\angle CAE + m\angle EAB + m\angle ABC + m\angle ACB$$

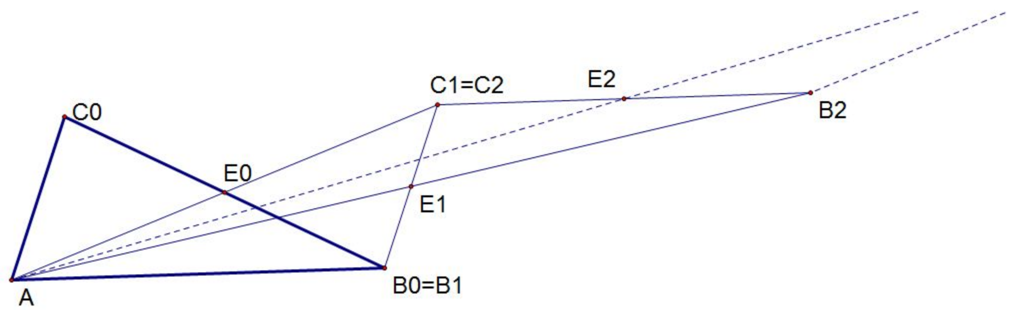
$$= m\angle BDE + m\angle EAB + m\angle ABE + m\angle DBE.$$



Thus we have the angle sum of ΔADB is equal to that of ΔABC, and by a similar argument ΔACD also has this same angle sum. Since either $m\angle DAB \leq \frac{1}{2}m\angle CAB$ or $m\angle DAC \leq \frac{1}{2}m\angle CAB$, we can set B_0 to B or C respectively, and we will have the triangle ΔAB_0D which has the same angle sum as ΔABC, and $m\angle DAB_0 = \frac{1}{2}m\angle BAC$. ∎

Theorem 2.33 The Saccheri-Legendre Theorem. *The angle sum of any triangle is less than or equal to* $180°$.

Proof: Assume there is ΔAB_0C_0 with angle sum $180° + p$, for p a positive real number. Then by Theorem 2.32 there is ΔAB_1C_1 with angle sum $180° + p$, and $m\angle C_1AB_1 \leq \frac{1}{2}m\angle C_0AB$, and further that there is C_2 and B_2 such that ΔAB_2C_2 has angle sum $180° + p$ and $m\angle C_2AB_2 \leq \frac{1}{2}m\angle C_1AB_1 \leq \frac{1}{4}m\angle C_0AB_0$.



We can continue this process, and by the Archimedian Principle there is $n \in \mathbb{N}$ such that n applications will give us $\Delta C_n A B_n$ with angle sum $180° + p$, and

$$m\angle C_n A B_n \leq \frac{1}{2^n} m\angle C_0 A B_0 \leq p.$$

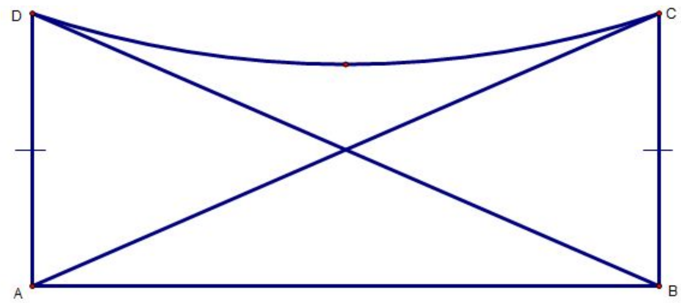
This implies that $m\angle AB_nC_n + m\angle AC_nB_n \geq 180°$. This contradicts Theorem 2.31, so no such triangle ΔAB_0C_0 may exist, and every triangle must have angle sum less than or equal to 180°. ∎

Corollary 2.2.2 *The angle sum of any quadrilateral is less than or equal to* 360°.

Theorem 2.34 *In a Saccheri quadrilateral:*

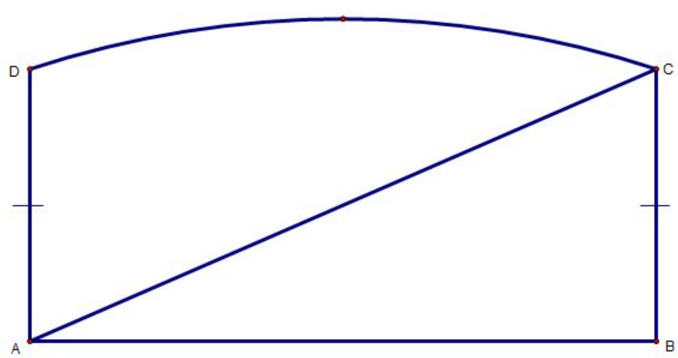
1. *The diagonals of a Saccheri quadrilateral are congruent.*
2. *The summit angles of a Saccheri quadrilateral are congruent.*
3. *The summit angles of a Saccheri quadrilateral are not obtuse.*
4. *The line joining the midpoints of the base and summit of a Saccheri quadrilateral are perpendicular to both.*
5. *The summit and base of a Saccheri quadrilateral are parallel.*

Proof: Given a Saccheri quadrilateral $\square ABCD$, for part (1), Axiom 15 gives us that $\Delta ABD \cong \Delta BAC$. Thus $\overline{AC} \cong \overline{BD}$.

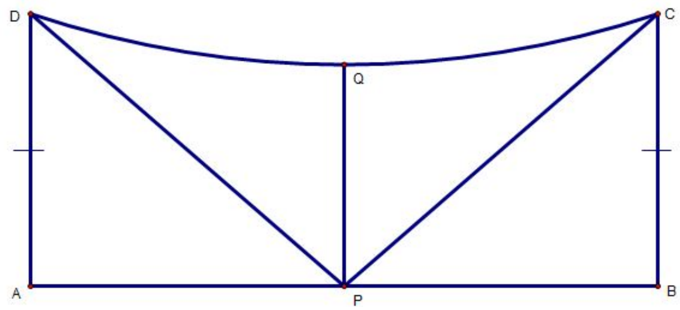


For part (2), we have, by hypothesis, that $\overline{AD} \cong \overline{BC}$ and that the base angles of $\square ABCD$ are right angles. With this, combined with part (1) and Side-Side-Side congruence, we have that $\Delta ACD \cong \Delta BDC$. Thus, $\angle ADC \cong \angle BCD$.

Now, for part (3) we'll assume that the summit angles of $\square ABCD$ are obtuse. By this assumption $\square ABCD$ must have angle sum greater than 360°. If this is the case, then one of ΔACD or ΔABC has an angle sum greater than 180°, which contradicts the Saccheri-Legrendre Theorem, so each summit angle must be right or acute.

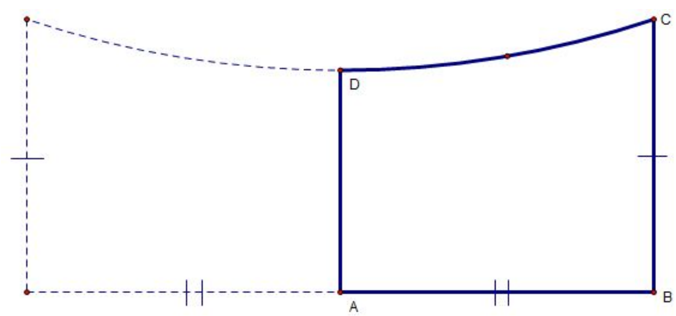


Lastly, we'll let P be the midpoint of $\overline{AB}$ and Q be the midpoint of $\overline{DC}$. By Axiom 15 $\Delta ADP \cong \Delta BCP$, so $\overline{DP} \cong \overline{CP}$. Then, by Side-Side-Side congruence, $\Delta DQP \cong \Delta CQP$, and so $\angle DQP \cong \angle CQP$. Since these are congruent supplementary angles, they are right angles and $\overline{QP}$ is perpendicular to $\overline{CD}$. A symmetric argument shows that $\overline{QP}$ is also perpendicular to $\overline{AB}$. From this and Theorem 2.19, part (5) should be clear. ∎

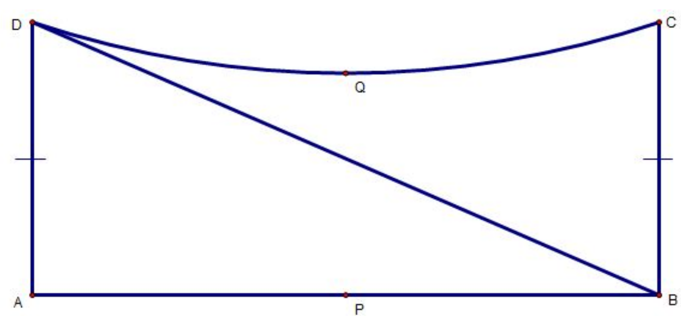


Corollary 2.2.3 *The fourth angle of a Lambert quadrilateral is not obtuse.*

Note: this is equivalent to saying that the summit angles of a Saccheri quadrilateral are not obtuse. In the figure below, we see that halving a Saccheri quadrilateral about the midpoints of its summit and base results in a Lambert quadrilateral.



Theorem 2.35 *The length of the summit of a Saccheri quadrilateral is greater than or equal to the length of the base.*



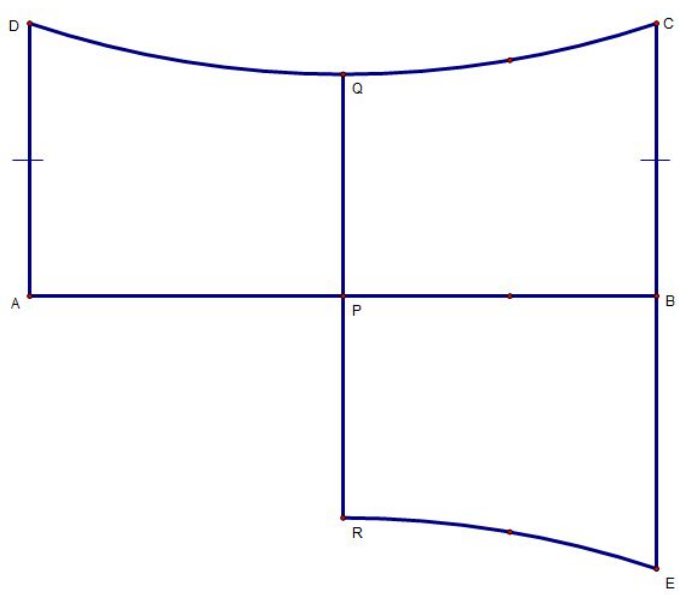
Proof: Given a Saccheri quadrilateral $\square ABCD$, if $\angle ADB \leq \angle CBD$, then $m\overline{AB} \leq m\overline{CD}$ by the Hinge Theorem (2.28). So for the sake of contradiction assume that $m\angle ADB > m\angle CBD$. Since $m\angle ABD + m\angle CBD = 90°$, we have

$$m\angle ABD + m\angle ADB > 90°,$$

giving ΔABD an angle sum greater than $180°$. This contradicts the Saccheri-Legendre Theorem, so $m\angle ADB \leq m\angle CBD$, and $m(\overline{CD}) \geq m(\overline{AB})$. ∎

Theorem 2.36 *The length of the segment joining the midpoints of a Saccheri quadrilateral is less than or equal to the length of it's sides.*

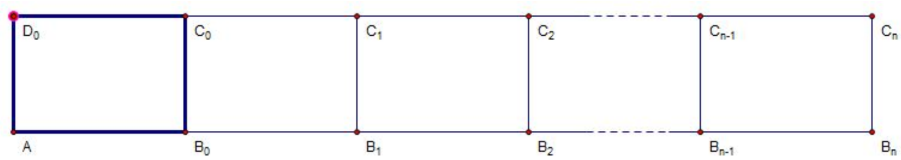
Note: This is equivalent to saying that the length of any side of a Lambert quadrilateral included by two right angles is less that or equal to the length of the opposite side.



Proof: Given a Saccheri quadrilateral $\square ABCD$, let P be the midpoint of $\overline{AB}$, and Q be the midpoint of $\overline{CD}$. Extend $\overline{QP}$ to R such that $Q - P - R$ and

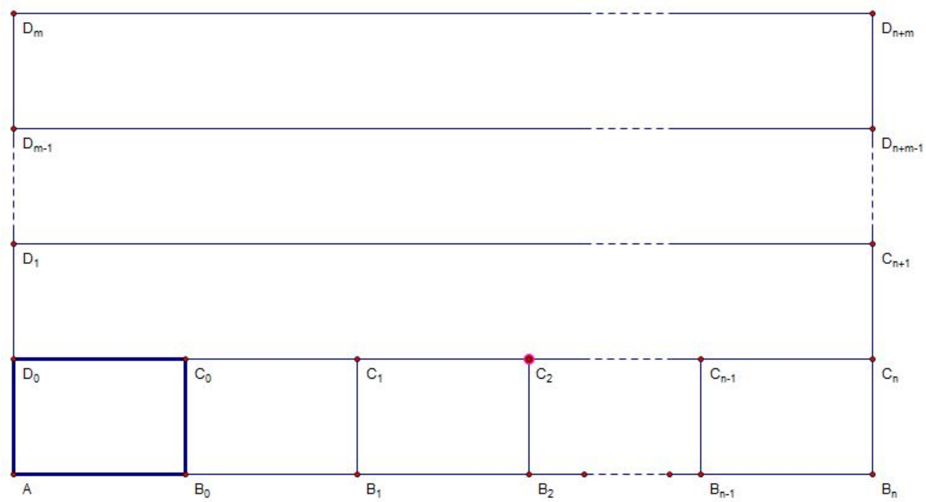
$\overline{QP} \cong \overline{PR}$. Also extend $\overline{CB}$ to E such that$C - B - E$ and $\overline{CB} \cong \overline{BE}$. By S-A-S-A-S congruence (Theorem 2.24), $\square PBCQ \cong \square PBER$. Notice, then, that $\square QREC$ is a Saccheri quadrilateral, so that by Theorem 2.35 $m(\overline{PQ}) = \frac{1}{2}m(\overline{QR}) \leq \frac{1}{2}m(\overline{CE}) = m(\overline{CB})$. This completes our proof. ∎

Theorem 2.37 *If one rectangle exists, then there exists a rectangle with two arbitrarily large sides.*



Proof: Assume that $\square AB_0C_0D_0$ is a rectangle. Given two positive numbers p and q, we want to prove the existence of a rectangle with one side length greater than p and another side length greater than q. Start by extending $\overline{AB_0}$ to B_1 and $\overline{D_0C_0}$ to C_1 so that $A - B_0 - B_1$ and $D_0 - C_0 - C_1$, and $\overline{AB_0} \cong \overline{B_0B_1}$ and $\overline{D_0C_0} \cong \overline{C_0C_1}$. We then have that $\square AB_0C_0D_0 \cong \square B_0B_1C_1C_0$ by S-A-S-A-S congruence, and that $\square AB_1C_1D_0$ is a rectangle with on side length $2m\overline{AB_0}$. By the Archimedean Principle there is $n \in \mathbb{N}$ such that after n iterations we will have a rectangle $\square AB_nC_nD_0$ in which $m\overline{AB_n} > p$.

Repeating this process for sides $\overline{AD_0}$ and $\overline{B_nC_n}$ (first extending $\overline{AD_0}$ to D_1 and $\overline{B_nC_n}$ to C_{n+1}), the same argument says that there is $m \in \mathbb{N}$ such that m iterations results in $\square AB_nC_{n+m}D_m$, where $m\overline{AD_m} > q$. ∎



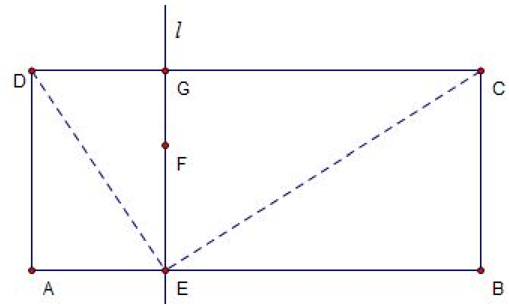
Theorem 2.38 *If a rectangle $\square ABCD$ exists, then for any point E such that $A - E - B$ the unique line l perpendicular to $\overline{AB}$ at E intersects $\overline{CD}$ at G such that*

1. *$D - G - C$,*

2. *$\square AEGD$ is a rectangle, and*

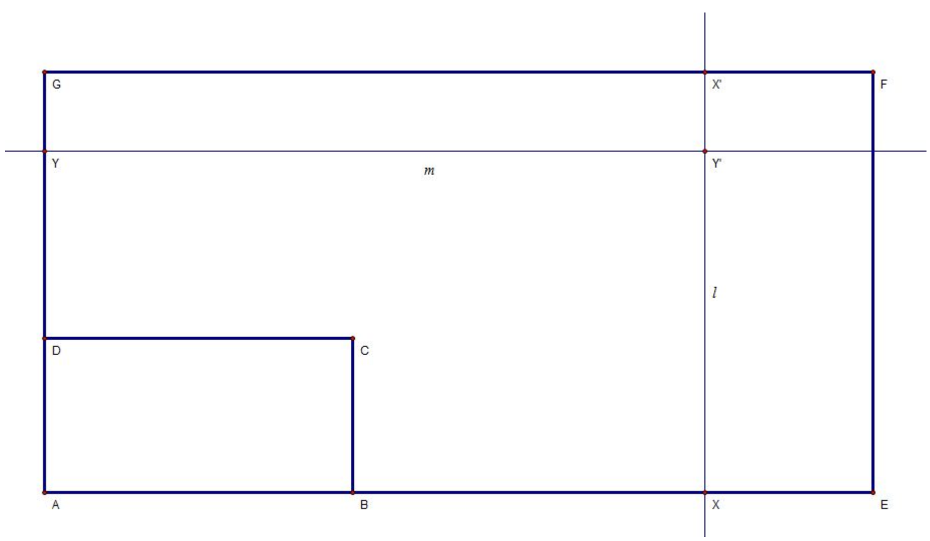
3. *$\square EBCG$ is a rectangle.*

Proof: Given a rectangle $\square ABCD$, let $A - E - B$. Notice that by the Saccheri-Legendre Theorem the angles $\angle CEB$ and $\angle DEA$ must be acute. Now let F be a point on the same side of $\overleftrightarrow{AB}$ as C and D such that $m\angle AEF = m\angle FEB = 90°$; note that F must then be interior to $\angle CED$ (otherwise either $\angle AEF$ or $\angle BEF$ would be acute). Then, considering the triangle ΔCDE, $\overleftrightarrow{EF}$ must intersect $\overline{CD}$ at some point G such that $C - G - D$ by the Crossbar Theorem. It follows from the Saccheri-Legendre Theorem that the angle sum of any quadrilateral is less than or equal to 360°. So it must be that $\angle DGE \cong \angle CGE$ are right angles. Thus, $\square AEGD$ and $\square EBCG$ are rectangles. ∎

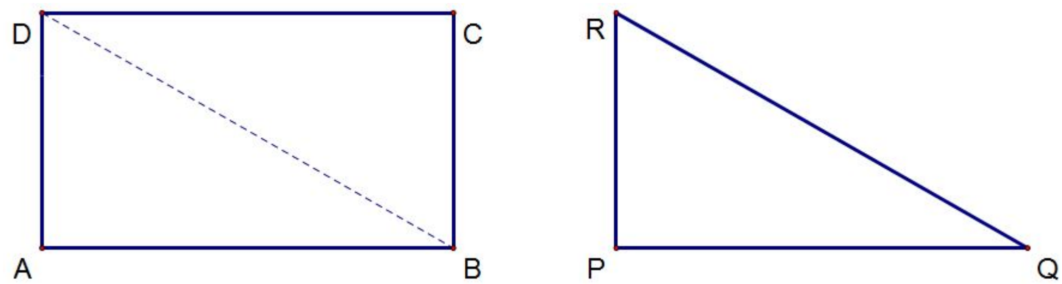


Theorem 2.39 *If one rectangle exists, then there exists a rectangle with two sides of any desired length.*

Proof: Assume that $\square ABCD$ is a rectangle, let p and q be two positive numbers, and let $\square AEFG$ be the rectangle guaranteed to exist by the previous theorem in which $m\overline{AE} > p$ and $m\overline{AG} > q$. There is X on $\overline{AE}$ such that $A - X - E$, and $m\overline{AX} = p$. Let l be perpendicular to $\overline{AE}$ through X. By the previous theorem, l intersects $\overline{GF}$ at a point X' such that $\square AXX'G$ and $\square XEFX'$ are rectangles. A symmetric argument gives us the point Y on $\overline{AG}$ such that $A - Y - G$ and $m\overline{AY} = q$, and that the line $m \perp \overline{AG}$ at Y gives us yet another rectangle $\square AXY'Y$ with the desired side lengths. ∎



Theorem 2.40 *If one rectangle exists, then every right triangle has an angle sum of* $180°$*.*



Proof: Assume a rectangle exists. Then given a triangle ΔPQR where $\angle RPQ$ is a right angle, the previous theorem guarantees the existence of a rectangle $\square ABCD$ where $\overline{AB} \cong \overline{PQ}$ and $\overline{AD} \cong \overline{PR}$. By S-A-S $\Delta ABD \cong \Delta PQR \cong \Delta CDB$. By these congruences and the fact that $\square ABCD$ has angle sum $360°$, ΔABD and ΔBCD have angle sum $180°$, which implies that ΔPQR has angle sum $180°$. ∎

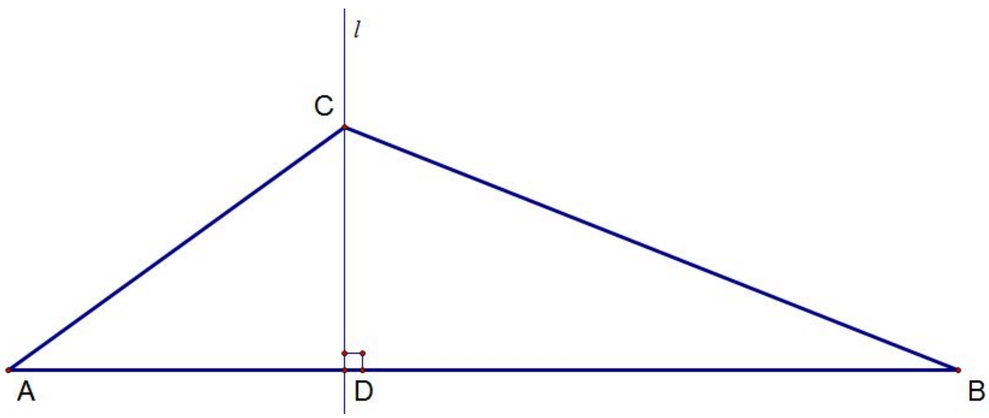
Theorem 2.41 *In a triangle* ΔABC*, if* $m\overline{AB} \geq m\overline{AC}$ *and* $m\overline{AB} \geq m\overline{BC}$*, then the line* l *perpendicular to* $\overleftrightarrow{AB}$ *passing through* C *intersects* $\overline{AB}$ *at* D *such that* $A-D-B$*.*

Proof: We'll prove this by contradiction, so first note that there are two alternatives to the statement above: either D is not distinct from A or B, or D does not lie on $\overline{AB}$ (ie: $D-A-B$ or $A-B-D$). For the first case, wlog we may assume D and A are the same point. Then $\overleftrightarrow{AC}$ and l are the same line; however, this implies that $\angle CAB$ is a right angle. We know that, in a triangle, a largest side if opposite the largest angle, so $m\angle ACB \geq 90$. This leads to ΔABC have two right angles, which contradicts earlier work.

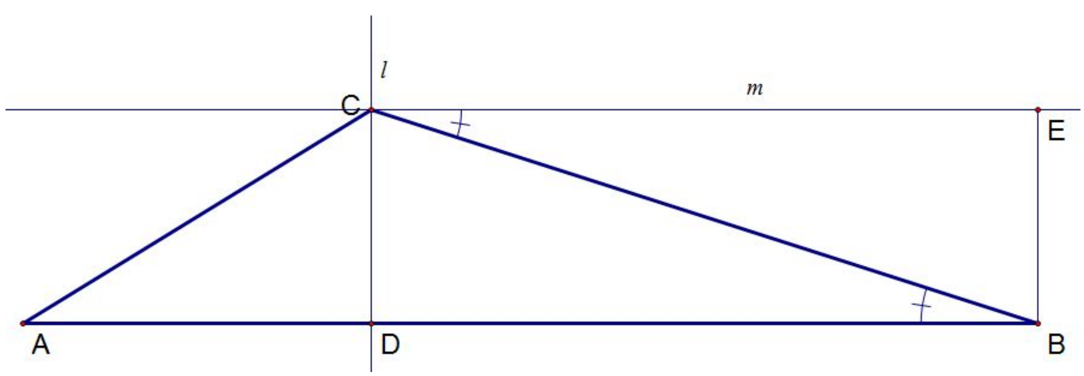
For the second case, assume that $D-A-B$. This gives us a new triangle ΔDAC with exterior angle $\angle CAB$. Since $m\angle CAB \geq m\angle ADC = 90$, for our hypothesis to hold both $\angle CAB$ and $\angle ACB$ must be obtuse. This implies ΔABC has angle sum greater than 180°, contradicting the Saccheri-Legendre Theorem. Therefore, it must be that $A - D - C$. ∎

Theorem 2.42 *If one rectangle exists, then every triangle has angle sum of* 180°*.*

Proof: Assume that a rectangle exists, and consider a triangle ΔABC where $m\overline{AB} \geq m\overline{AC}$ and $m\overline{AB} \geq m\overline{BC}$. Let l be perpendicular to $\overline{AB}$ through C, and intersection $\overline{AB}$ at D, where $A-D-B$. Then ΔADC and ΔCBD are right triangles, and therefore have angle sums of 180°. Then, because the angle sum of ΔABC is $m\angle DBC + m\angle BCD + m\angle DCA + m\angle CAD$, and $m\angle DAC + m\angle DCA = 90°$ and $m\angle DBC + m\angle CBD = 90°$, we have that the angle sum of ΔABC is 180°. ∎



Theorem 2.43 *If one triangle has angle sum of* 180°*, then a rectangle exists.*



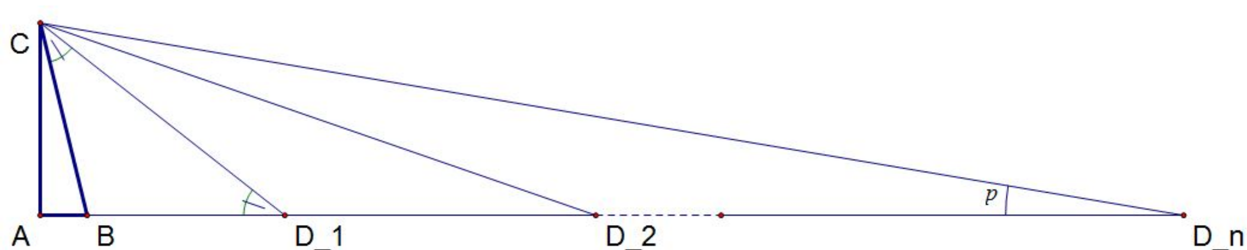
Proof: Suppose that ΔABC has angle sum $180°$, letting $\overline{AB}$ have length greater than or equal to the length of each of the remaining sides. Then the line l perpendicular to $\overline{AB}$ passing through C intersects $\overline{AB}$ at a point D such that $A - D - B$. As no triangle may have angle sum greater than $180°$, along with the assumption that ΔABC has angle sum $180°$, it must be that ΔADC and ΔCDB have angle sums $180°$.

Now, let m be the line perpendicular to $\overline{CD}$ through C, and take point E to be on the same side of l as B such that $\overline{CE} \cong \overline{DB}$. At this point, note that because ΔDBC has angle sum $180°$ and $\angle CDB$ is a right angle, $\angle DBC$ and $\angle DCB$ are complementary. Therefore, $\angle DBC \cong \angle BCE$. Then, by S-A-S, we have congruent triangles $\Delta DBC \cong \Delta ECB$, so $m\angle DBC + m\angle CBE = m\angle DCB + m\angle BCE = 90°$, and we have a quadrilateral with four right angles; we have the existence of a rectangle $\square DBEC$. ∎

Theorem 2.44 *If one triangle has an angle sum of* $180°$*, then every triangle has angle sum of* $180°$.

This follows immediately from Theorems 2.33 and 2.32.

Theorem 2.45 *Given any triangle* ΔABC *in which* $\angle CAB$ *is a right angle, and given any* q *where* $0 < q < 90$*, there exists* ΔADC *where* $\angle CAD$ *is a right angle, and* $m\angle ADC < q$.



Proof: Given a right triangle ΔABC, and letting q be the desired angle measure, $0 < q < 90$, we may take D_1 on $\overleftrightarrow{AB}$ such that $A - B - D_1$ and $\overline{CB} \cong \overline{BD_1}$. By the Isosceles Triangle Theorem $\angle BCD_1 \cong \angle BD_1C$. Also, we know that $m\angle CBD_1 + m\angle CD_1B + m\angle D_1CB \leq 180°$, but $m\angle CBD_1 = 180° - m\angle CBA$. Thus, using this and the stated angle congruence, $180° - m\angle CBA + 2m\angle CD_1B \leq 180°$, or $m\angle CD_1B \leq \frac{1}{2}m\angle CBA$. Now, there is $n \in \mathbb{N}$ such that n iterations of this process gives us the right triangle ΔAD_nC in which $m\angle AD_nC = \frac{1}{2^n}m\angle ABC = p < q$. ∎

2.3 Equivalencies to Euclid V

In future chapters we will assume a denial of Euclid V. So, in the interest of knowing what else we will be denying, some equivalencies of Euclid V follow. We'll repeatedly use some of the more recent result from the previous section. To summarize, we have

There exists a right triangle with angle sum 180°	$\Leftrightarrow$	Every triangle has angle sum 180°
	$\Leftrightarrow$	A rectangle exists
	$\Leftrightarrow$	There exists a triangle with angle sum 180°

Theorem 2.46 *Euclid's fifth postulate is equivalent to the Euclidean Parallel Postulate.*

Proof: $\Rightarrow$ Assume Euclid V, and let l be a line and P a point not on l. Then there is a unique line r passing through P such that l and r are perpendicular. Also, there is a unique line s passing through P such that r and s are perpendicular. By Euclid V, s and l are parallel. Now, letting t be any line passing through P and distinct from r and s, we have that the interior angles on one side of r, given by the intersections of r with t and with l, are together less than $180°$ by Axiom 12. Euclid V then says that t and l must intersect. Thus, s is unique as a line passing through P and parallel to l, giving the Euclidean Parallel Postulate.

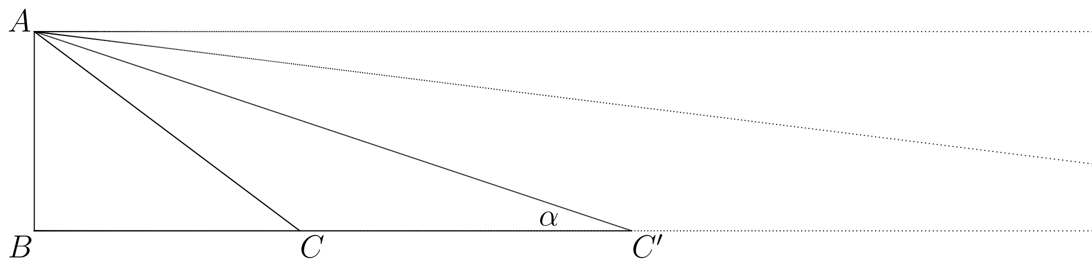
$\Leftarrow$ Assume the Euclidean Parallel Postulate, and let l be a line and P a point not on l. Then there is a unique line s which passes through P and is parallel to l. Again, taking r to be the unique line perpendicular to l and passing through P, and letting t be the line perpendicular to r and passing through P, we have that t is parallel to l. Then by the Euclidean Parallel Postulate it must be that t and s are the same line. Now letting u be any line passing through P, distinct from s, it must be that u intersects l at some point. If u is distinct from r, then there is one side of r on which the intersections of r with u and l create interior angles whose sum is less that two right angles. By the Saccheri-Legendre Theorem (2.33), u and l must intersect on the side of r giving the interior angles with sum less than two right angles. We therefore have Euclid V. ∎

Theorem 2.47 *Euclid V is equivalent to Proclus' Axiom: if two lines are parallel, and another line passes through one of these, then it also passes through the other.*

Proof: $\Rightarrow$ Assume Euclid V, and let l and t be parallel lines. Then for any point P on t, t is the only line parallel to l passing through P by the previous theorem. Therefore, any line r which intersects t (and is distinct from t) must also intersect l. By symmetry, any line which intersects l must also intersect t. Thus we have Proclus' Axiom.

$\Leftarrow$ Assume Proclus' Axiom, and take a line l and a point P. There exists at least one line t passing through P and parallel to l. By Proclus' Axiom any line r (distinct from t) intersecting t through P must also intersect l. Thus t is the only line passing through P, a point not on l, which is parallel to l. As l may be any line and P any point not on that line, we have the Euclidean Parallel Postulate, and therefore Euclid V. ∎

Theorem 2.48 *Euclid's fifth postulate is equivalent to every triangle having an angle sum of* 180°.



Proof: $\Leftarrow$ Assume that ΔABC is a right triangle with angle sum 180°. Then, letting $\angle ABC$ be the right angle of our triangle, we have that for any α, where $0 < \alpha < 90$, we may extend $\overline{BC}$ to C_0 such that $m\angle BC'A = \beta < \alpha$ by Theorem 2.45. Then there is a unique ray $\overrightarrow{AD}$ where $m\angle BAD = 180 - \alpha$ and D is interior to $\angle BAC_0$. By the Crossbar Theorem, $\overrightarrow{AD}$ must intersect $\overline{BC_0}$ at a point C'. Since ΔABC has angle sum 180°, every triangle has angle sum 180°, so it must be that $m\angle ABC' = \alpha$.

Then as $\alpha \to 0$, $m\angle BC'A \to 90$. This means that, for any ray $\overrightarrow{AD}$ with D on the same side of $\overleftrightarrow{AB}$ as C, if $m\angle BAD < 90°$, then $\overrightarrow{AD}$ will intersect $\overleftrightarrow{BC}$. By a symmetric argument, if a point E were on the opposite side of $\overleftrightarrow{AB}$ as C and if $m\angle BAE < 90°$, the $\overrightarrow{AE}$ will intersect $\overleftrightarrow{BC}$. Thus, a line l passing through A is parallel to $\overleftrightarrow{BC}$ only if it is perpendicular to $\overline{AB}$. This gives us the Euclidean Parallel Postulate, and hence Euclid V.

$\Rightarrow$ Now assume Euclid V, and take distinct lines l, m and n, such that $l \perp m \perp n$. Let P a point of n not on m, and let k be the line through P perpendicular to n. If k is not perpendicular to l, then there is another line, k' distinct from k, which passes through P and is perpendicular to l. However, by construction both k and k' are parallel to m and pass through P, and by the previous theorem this is impossible. So k' and k must be the same line. This gives us the existence of a rectangle, formed the the intersection of line k, l, m, and n, and the existence of a rectangle is equivalent to the desired result. ∎

Corollary 2.3.1 *Euclid's fifth postulate is equivalent to the existence of a quadrilateral with angle sum* $360°$*.*

Lemma 2.3.1 *If each interior angle of a quadrilateral* $\square ABCD$ *is a right angle, then opposite sides of* $\square ABCD$ *are congruent.*

Proof: Suppose that each angle of $\square ABCD$ is a right angle; note that Euclid V follows since we have a quadrilateral with angle sum $360°$, so that each of ΔABD and ΔCDB must have angle sums of $180°$. Then we have the following equations:

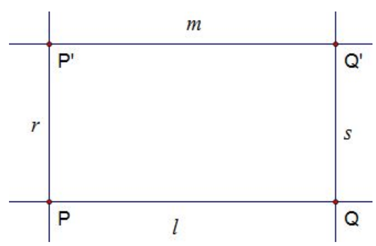
$$\begin{aligned} m\angle ABD + m\angle DBC &= 90° \\ m\angle ADB + m\angle BDC &= 90° \\ m\angle ABD + m\angle ADB &= 90° \\ m\angle CBD + m\angle CDB &= 90°. \end{aligned}$$

This allows us to say that $\angle ABD \cong \angle CDB$ and $\angle ADB \cong \angle CBD$. Since $\overline{BD}$ is congruent to $\overline{DB}$, axiom 15 gives us that $\Delta ABD \cong \Delta CDB$, so that $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$. ∎

Theorem 2.49 *Euclid V is equivalent to parallels being everywhere equidistant.*

Proof: $\Rightarrow$ Assume Euclid V, and let l and m be parallel lines. Take r to be perpendicular to l through a point P on l, and note that r must intersect m

through a point P' by Proclus' Axiom. As a line perpendicular to r through P' will be parallel to l, and because we have the Euclidean Parallel Postulate, m must be perpendicular to r. Now let Q be a point on l distinct from P, and let s be a line perpendicular to l through Q. Then just as r must be perpendicular to m, s must be perpendicular to m through some point Q'. This gives us a rectangle $\square PP'Q'Q$. This holds for all lines s distinct from r, so regardless of the length $m\overline{PQ}$ we will have $\overline{PP'} \cong \overline{QQ'}$. Since congruent segments have the same measure, l and m are everywhere equidistant.



$\Leftarrow$ Assume now that parallel lines are everywhere equidistant. Then given two parallel lines, l and m, we may choose any two distinct points of l, P and Q, and drop line r and s perpendicular to m passing through P and Q respectively. Letting P' be the points at which r intersects m, and Q' be the points at which s intersects m, by hypothesis we have that $\overline{PP'} \cong \overline{QQ'}$. This gives us a Saccheri quadrilateral $\square PQQ'P$, and by Theorem 2.34 $\angle PP'Q' \cong \angle QQ'P'$.

Since l is a common perpendicular between r and s, r and s must be parallel. Then the hypothesis again allows us to say that $\overline{PQ} \cong \overline{P'Q'}$. Then, considering triangles $\Delta PP'Q'$ and $\Delta P'PQ$, by S-S-S congruence we know that $\Delta PP'Q' \cong \Delta P'PQ$, so $\angle P'PQ \cong \angle PP'Q' \cong \angle QQ'P'$. Therefore, each angle of $\square PP'Q'Q$ is a right angle, and opposite sides are congruent, so $\square PP'Q'Q$ is a rectangle. Since parallel lines being everywhere equidistant implies a rectangle exists, it also implies Euclid V by Corollary 2.3.1. ∎

The next equivalence we will consider is that between Euclid V and the converse of the Alternate Interior Angle Theorem:

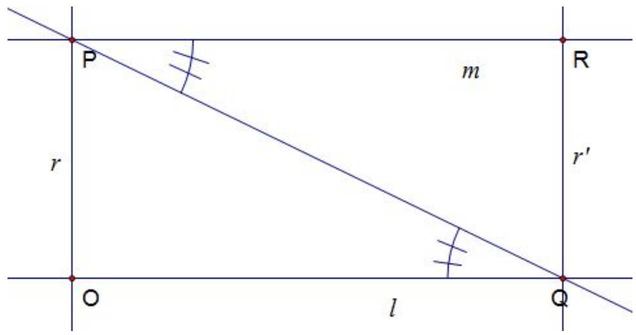
> *If two lines l and m are parallel, and r is a transversal of l and m, then the alternate interior angles formed by the intersection of r with l and of r with m are congruent.*

Theorem 2.50 *Euclid V is equivalent to the converse of the Alternate Interior Angle Theorem.*

Proof: $\Rightarrow$ Assume Euclid V, so that every triangle has angle sum $180°$, and consider parallel lines l and m. Let s be a transversal of these lines, intersecting m at P and l and Q. Note that if s is perpendicular to l, then s is also perpendicular to m since there exists exactly one line parallel to l passing through P, and if a common perpendicular exists between two lines then they are parallel (ie: if $s \perp l$ and $s \not\perp m$, then a second line parallel to l could be constructed which is perpendicular to s at P, but m must be the unique line parallel to l through P, so $s \perp l$ implies $s \perp m$). Since right angles are congruent, our conclusion is satisfied if $s \perp l$.

If, instead, s is not perpendicular to l, then s is also not perpendicular to m; assume this is the case. Then there exists a line r which passes through P and is perpendicular to l (and therefore m). Letting O be the point at which r intersects l, it follows that $\angle OPQ$ and $\angle OQP$ are complementary. Similarly, there exists a line r' which passes through Q and is perpendicular to both l and m. Letting R be the intersection of r' with m, we have that $\angle RPQ$ and $\angle RQP$ are also complementary. Notice that we also have complementary pairs of angles $\angle OQP$ and $\angle RQP$, and $\angle OPQ$ and $\angle RPO$. Since $\angle OQP$ and $\angle RPQ$ are both complementary to angle $\angle RQP$, it follows from Theorem 2.12 that $\angle OQP \cong \angle RPQ$.

Therefore, the converse of the Alternate Interior Angle Theorem follows from Euclid V.



$\Leftarrow$ Assume the converse of the Alternate Interior Angle Theorem, and take l and m to be parallel lines. Then if r is perpendicular to l at a point P, it is also perpendicular to m at a point Q. Let s be yet another line, distinct from m and passing through Q, which intersects l at some point R. Then $\angle PQR$ and $\angle PRQ$ are complementary since s forms congruent alternate interior angles by its intersections with l and m, and we therefore have a triangle ΔPQR with angle sum $180°$. Since Euclid V is equivalent to a triangle have angle sum $180°$, we then have that the converse of the Alternate Interior Angle Theorem implies Euclid V. ∎

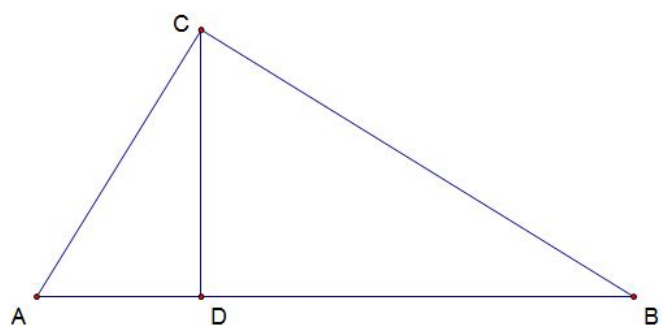
Corollary 2.3.2 *In Euclidean Geometry, given distinct lines k, l, m, and n, if $k \parallel l$, $k \perp m$ and $l \perp n$, then $m \parallel n$.*

Proof: Given our hypothesis, let P be the point of intersection of k and m, and Q be the point of intersection of l and n. Since l and n pass through Q, and l is the unique line parallel to k passing through Q, n must intersect k; let R be the point at which k intersects n. Similarly, l and m must intersect at some point S. By Theorem 2.50, alternate interior angles formed by the intersections of n with k and l are congruent; however, $n \perp l$, so $\angle QRS$ is a right angle ($k \perp n$). Since m and n

are distinct lines for which there is a common perpendicular, k, m and n are parallel by Theorem 2.19. ∎

Theorem 2.51 *Euclid V is equivalent to the Pythagorean Theorem.*

Proof: $\Rightarrow$ Assume Euclid V, and take a right triangle ΔABC where $\angle BCA$ is a right angle; this triangle must have angle sum 180° by Theorem 2.3.3. Taking a point D on $\overline{AB}$ such that $A - D - B$ and $\overline{CD} \perp \overline{AB}$, we have two triangles ΔADC and ΔBCD. Then $\angle DCA \cong \angle DBC$ and $\angle DCB \cong \angle DAC$, so we have $\Delta ADC \sim \Delta CDB \sim \Delta ACB$. Then we have $\frac{|AC|}{|AB|} = \frac{|AD|}{|AC|}$ and $\frac{|BC|}{|AB|} = \frac{|DB|}{|BC|}$. This implies that $|AC|^2 = |AD||AB|$ and $|BC|^2 = |AB||DB|$, which in turn implies that $|AC|^2 + |BC|^2 = |AB|(|AD| + |DB|) = |AB|^2$. Thus we have the Pythagorean Theorem.



$\Leftarrow$ Assume the Pythagorean Theorem, and let ΔABC be an isosceles right triangle, with $\angle CBA$ a right angle, and $\overline{AB} \cong \overline{BC}$. Then letting D be the midpoint of $\overline{AC}$, we have congruent triangle $\Delta DBA \cong \Delta DBC$. From this, we have $\angle ADB \cong \angle CDB$. As these angles are also supplementary, we have that $\overline{BD} \perp \overline{AC}$. So

$$\begin{aligned} |AB|^2 + |BC|^2 &= |AC|^2, \\ |CD|^2 + |DB|^2 &= |BC|^2, \text{ and} \\ |AD|^2 + |DB|^2 &= |AB|^2. \end{aligned}$$

This leads us to

$$
\begin{aligned}
|CD|^2 + |AD|^2 + 2|DB|^2 &= |AB|^2 + |BC|^2 \\
&= |AC|^2 \\
&= (|AD| + |DC|)^2 \\
&= |CD|^2 + 2|CD||DA| + |DA|^2 \\
2|DB|^2 &= 2|CD||DA| \\
|DB|^2 &= |CD|^2
\end{aligned}
$$

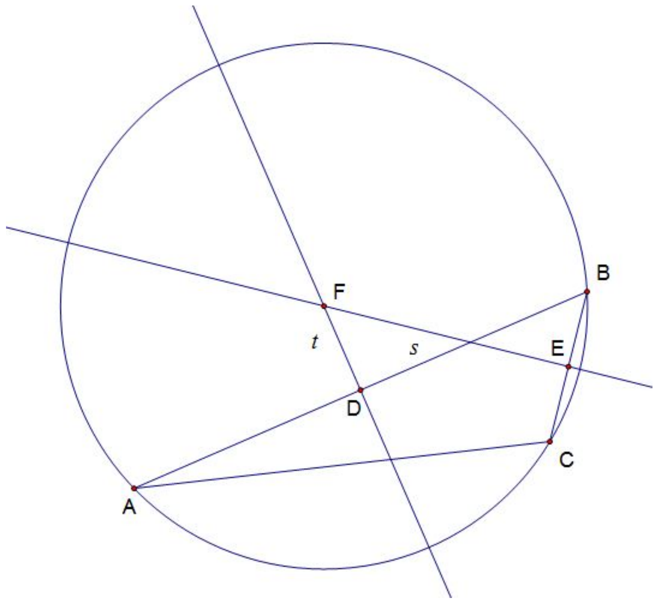
Therefore, $|CD| = |DA| = |DB|$, and we have that $\Delta CDB \cong \Delta ADB$ are congruent isosceles right triangles, so

$$90 = m\angle ABC = m\angle DBC + m\angle DBA = m\angle DCB + m\angle DAB,$$

and ΔABC has an angle sum of 180. Thus we have Euclid V. ∎

Theorem 2.52 *Euclid V is equivalent to the following statement: given a triangle ΔABC, a circle γ may be constructed passing through A, B, and C.*

Proof: $\Rightarrow$ Assume Euclid V, and consider a triangle ΔABC. Let t and s be perpendicular bisectors of the segments $\overline{AB}$ and $\overline{BC}$ respectively. Let D and E be the midpoints of segments $\overline{AB}$ and $\overline{BC}$ respectively. Then t and s intersect at some point F; if $t \parallel s$, then by corollary 2.3.2 $\overleftrightarrow{AB} \parallel \overleftrightarrow{BC}$, but $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ clearly intersect at B. F is equidistant from A, B, and C by Theorem 2.17, so a circle γ may be constructed with center F and radius $|FA|$ which will pass through A, B, and C.



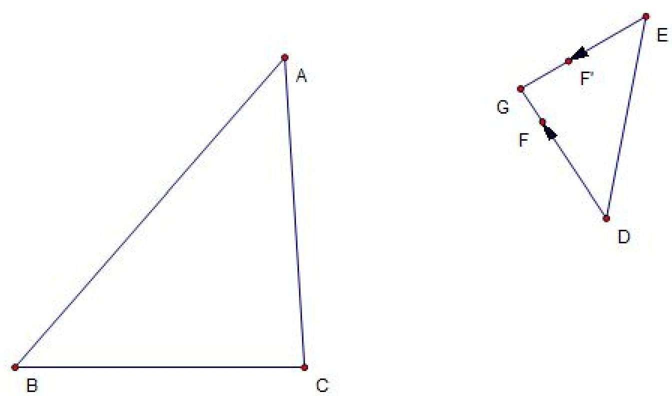
$\Leftarrow$ Assume that any triangle may be circumscribed and let ΔABC be an isosceles right triangle, with $\angle CAB$ the right angle. Also let γ be the circle circumscribed about ΔABC, then $\overline{BC}$ is a diameter of γ. Let O be the center of γ, so that we have triangles ΔOAB and ΔOAC. Since $\overline{OA} \cong \overline{OB} \cong \overline{OC}$ and $\overline{AB} \cong \overline{AC}$, we have by S-S-S that $\Delta OAC \cong \Delta OAB$, and that each is an isosceles triangle. However, O must be the midpoint of $\overline{BC}$ since $\overline{BC}$ is a diameter of γ, so $\angle BOA$ and $\angle COA$ are both supplementary and congruent, and therefore right angles. By the given triangle congruence, $\angle OCA \cong \angle OAC \cong \angle OAB \cong \angle OBA$. Since $\angle OAC$ and $\angle OAB$ are complementary, $\angle OCA$ and $\angle OBA$ are also complementary bay the stated angle congruences. Therefore, ΔABC has angle sum $180°$. ∎

Theorem 2.53 *Euclid V is equivalent to Wallace's Postulate: Given any triangle ΔABC and given any segment $\overline{DE}$, there exists a triangle $\Delta DEF \sim \Delta ABC$. Note that $\overline{AB}$ need not be congruent to $\overline{DE}$, so Euclid V is equivalent to the existence of similar, but not congruent, triangles.*

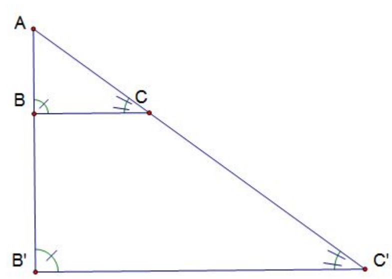
Proof: $\Rightarrow$ Assume Euclid V so that every triangle has angle sum 180°. Considering a triangle ΔABC and a segment, $\overline{DE}$, there exists F such that $\angle EDF \cong \angle BAC$. There also exists a point F', on the same side of $\overleftrightarrow{DE}$ as F, such that $\angle DEF' \cong \angle ABC$. Since ΔABC has angle sum 180°, it must be that $m\angle ABC + m\angle BAC < 180°$. Then, by Euclid V, $\overrightarrow{EF'}$ and $\overrightarrow{DF}$ must intersect at a point G on the same side of $\overleftrightarrow{DE}$ as F. Also by Euclid V,

$$m\angle DGE = 180° - m\angle GDE - m\angle GED = 180° - m\angle CAB - m\angle CBA = m\angle ACB,$$

so $\angle ACB \cong \angle DGE$. Therefore, $\Delta ABC \sim \Delta DEG$.



$\Leftarrow$ Assuming Wallace's postulate, there exist triangles ΔABC and $\Delta AB'C'$ where $A - B - B'$, $A - C - C'$, and $\Delta ABC \sim \Delta AB'C'$. Then we have that $\angle AB'C' \cong \angle ABC$, $\angle AC'B' \cong \angle ACB$, and $\angle CAB \cong \angle C'AB'$. Since $\angle B'BC$ is supplementary to $\angle ABC$ and $\angle C'CB$ is supplementary to $\angle ACB$, it follows from our congruences that $\square BB'C'C$ has angle sum 360°, which implies Euclid V. ∎

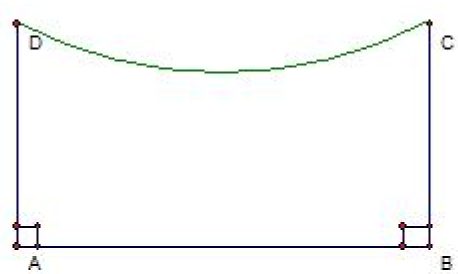


Chapter 3

Neutral Geometry and the Hyperbolic Axiom

3.1 Saccheri's Approach

Now we can discuss Saccheri's attempt at proving Euclid V in a little more depth. Recall that Saccheri started with a quadrilateral, $\square ABCD$ in which $\angle DAB \cong \angle ABC$ are right angles, and $\overline{DA} \cong \overline{BC}$,



and, from the following possibilities, hoped to eliminate the first and third:

1. The angles $\angle BCD \cong \angle CDA$ are acute angles.

2. The angles $\angle BCD \cong \angle CDA$ are right angles.

3. The angles $\angle BCD \cong \angle CDA$ are obtuse angles.

3.2 The Hyperbolic Axiom

We've seen several postulates and axioms, and now we'll visit another: the Hyperbolic Axiom.

The Hyperbolic Axiom: There exists a line l and a point P not on l, such that at least two distinct lines parallel to l pass through P.

When including this into our original 15 accepted axioms, we have the axioms for Hyperbolic Geometry. From previous work, we immediately have several theorems of Hyperbolic Geometry.

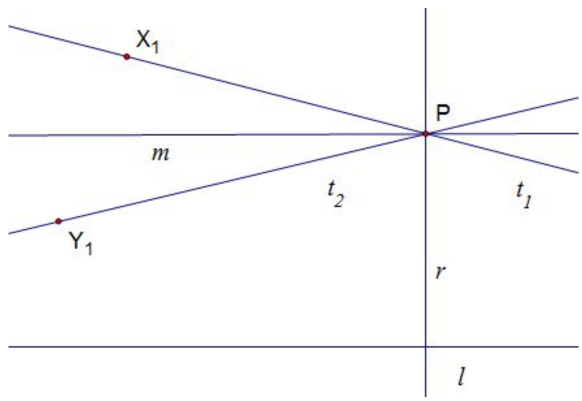
Theorem 3.1 *There exists a triangle with angle sum less than* $180°$*, and therefore all triangles have angle sum less than* $180°$*.*

Theorem 3.2 *No rectangle exists. Equivalently, any two parallel lines have at most one common perpendicular.*

Theorem 3.3 *The converse of the Alternate Interior Angle Theorem does not hold.*

Theorem 3.4 *Given a line l and a point P not on l, if two distinct lines, t_1 and t_2, are parallel to l and pass through P, then every line between t_1 and t_2 is parallel to l.*

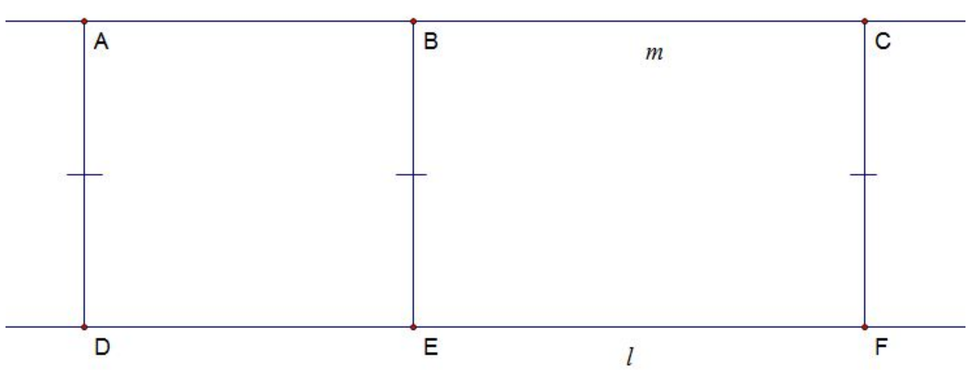
Proof: Take l, P, t_1, and t_2 as stated, and let r be perpendicular to l and pass through P. Take X_1 and Y_1 to be points of t_1 on opposite sides of r, and take X_2 and Y_2 to be points of t_2 on opposite sides of r, with X_1 and X_2 on the same side of r. A line m passing through P is between t_1 and t_2 if every point of m on the same side of r as X_1 is interior to angle $\angle X_1PY_1$, and every point of m on the same side of r as X_2 is interior to angle $\angle X_2PY_2$. Since t_1 and t_1 are parallel to l, no point of l is interior to $\angle X_1PY_1$ or interior to $\angle X_2PY_2$, so no point of a line m between t_1 and t_2 may also be a point of l; m is parallel to l. ∎



Lemma 3.2.1 *The summit angles of a Saccheri quadrilateral are acute.*

Proof: Recall that Euclid V is equivalent to the existence of a rectangle, and that the summit angles of a Saccheri quadrilateral are right or acute. Then by accepting the Hyperbolic Axiom we force the summit angles of a Saccheri quadrilateral to be acute. ∎

Theorem 3.5 *Given parallel lines l and m, no three points of one line are equidistant from the second; hence, parallel lines are not everywhere equidistant.*



Proof: Our approach here will be to assume that there exist three points on a given line which are equidistant from a second, parallel, line, and run into a contradiction. So take line l and m as described in the figure above, with $l \parallel m$ and three points of m, A, B, and C, equidistant from the line l. This results in two Saccheri quadrilaterals, $\square ABDE$ and $\square BCFE$ where $\angle ADE$, $\angle DEB$, $\angle BEF$, and $\angle EFC$ are right angles. Then $\angle DAB \cong \angle ABE$ and $\angle EBC \cong \angle BCF$. However, $\angle ABE$ and $\angle EBC$ are supplementary, so cannot both be acute (recall that the summit angles of a Saccheri quadrilateral must be acute). This gives us the desired contradiction, so the line m cannot be everywhere equidistant from the parallel line l. ∎

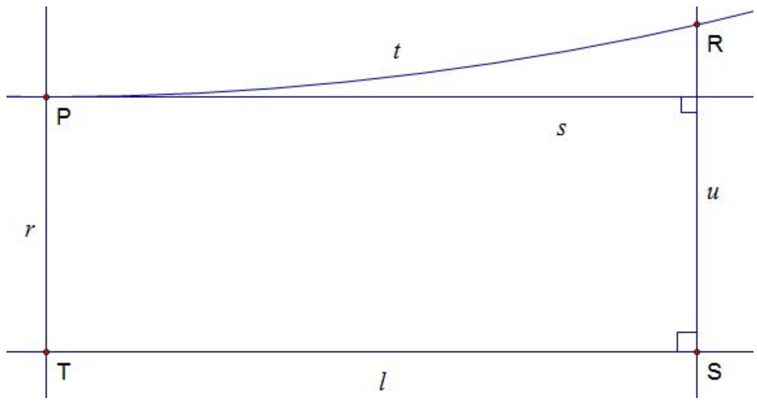
Corollary 3.2.1 *Given two lines, l and m, if l and m are parallel then either*

1. *there exist a pair of points on m equidistant from l or*

2. *there exist no two points of m which are equidistant from l.*

Theorem 3.6 *For every line l and every point P not on l, there exist distinct lines t and s, passing through P, which are parallel to l.*

To prove this, first consider line l and any point P not on l. Letting r be the line perpendicular to l through P, and letting t be perpendicular to r through P, gives us one line parallel to l passing through P. Now, letting R be a point of t distinct from P, take a line u to be perpendicular to l passing through R; this yields a Lambert quadrilateral $\square PTSR$, where T is the intersection of r and l and S is the intersection of u and l. From Neutral Geometry we have that $\angle SRP$ is either right or acute, but because we've taken on the Hyperbolic Axiom, a negation of Euclid V, $\angle SRP$ must be acute. There then exists a line s distinct from t which is perpendicular to u, passing through P. Since u is a common perpendicular to s and l, $l \parallel s$. We now have two lines, t and s, which are both parallel to l and pass through P, completing the proof.



Theorem 3.7 *Parallelism is not a transitive relation.*

This is a direct result of the Hyperbolic axiom; as we're given a line l, and two lines t and s which are parallel to l, yet intersect at a point P, we have that t is parallel to l, and l is parallel to s, but s is not parallel to t. This does not mean that no two parallel lines have a common parallel; rather, that there exist triples of lines where two of the lines are parallel to the third, but are not parallel to each other.

These few theorems of Hyperbolic Geometry should give the reader an idea of how different the Euclidean and Hyperbolic Geometries appear to be. More theorems will be considered later, but first we'll address the question of consistency of Hyperbolic Geometry. This question may be answered through the creation of a model which adheres to the axioms of Hyperbolic Geometry. There are several models of Hyperbolic geometry,a few which we'll consider later and one which we will focus on in chapter 7, but to begin we'll look at what's known as the Upper-Half Plane model.

3.3 A Model for Hyperbolic Geometry

The Upper-Half Plane model for Hyperbolic Geometry is often credited to Poincaré, though Stillwell refers to it as the Liouville-Beltrami model in *Sources of Hyperbolic Geometry* [4]. This model, and the others we see later, is embedded within a model of Euclidean Geometry. In this case, we'll be working in a subset of $\mathbb{C}$; specifically, the Upper-Half plane will be denoted $\mathbb{H}$, where $\mathbb{H} = \{z \in \mathbb{C} : z = x + iy,\ y > 0\}$. The line $L = \{z \in \mathbb{C} : z = x + i0\}$ is the boundary of $\mathbb{H}$, any point of which is referred to as an *ideal point*.

In $\mathbb{H}$, lines will appear either as the intersection of $\mathbb{H}$ with a Euclidean circle whose center lies on L, or as a Euclidean ray which is perpendicular to L. This may feel unnatural, but remember that we're building a model for Hyperbolic Geometry, so we will need this model to satisfy all the necessary axioms. At the same time, because $\mathbb{H}$ is embedded within a Euclidean model, we may use Euclidean ideas to build up and show consistency of the model. For example, the measure of an angle between two hyperbolic lines in $\mathbb{H}$ will be the same as the angle between the Euclidean semicircles, or Euclidean semicircle and Euclidean ray, that these Hyperbolic lines appear as. Distance, on the other hand, cannot be measured in such a Euclidean fashion; however, we can use Euclidean distance to help define distance in $\mathbb{H}$. This and more will be seen very soon, but first we'll need to consider some Euclidean results to help us define $\mathbb{H}$, and show that it is in fact a model for Hyperbolic Geometry.

Chapter 4

Necessities for Our Models

4.1 Concerning Circles

We have several ideas to visit before our first model of Hyperbolic Geometry can truly be considered. The first to attack is Euclidean results concerning circles. These will be important, not only to the basic layout of our first model, but also in showing the consistency of Hyperbolic Geometry itself through our first model. Throughout this section, we will be working in Euclidean Geometry, so Euclid V, and the statements we've shown to be equivalent in previous sections, will be assumed.

Remark 1 *Given two non-parallel chords of a circle, the center of the circle is the point of intersection of the perpendicular bisectors of the chords.*

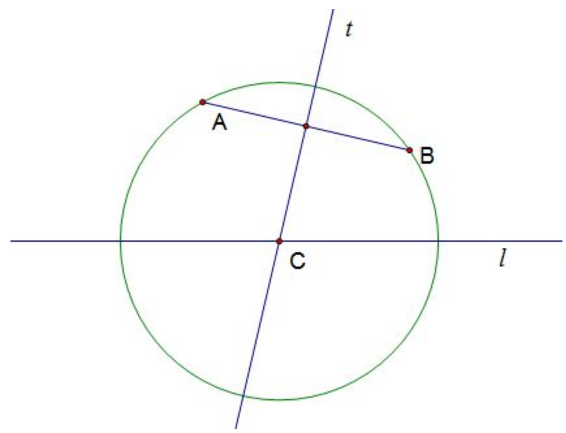
Theorem 4.1 *Through any three non-collinear points there is a unique circle.*

Proof: Let A, B, and C be three noncollinear points, with t_1 the perpendicular bisector of $\overline{AB}$ and t_2 the perpendicular bisector of $\overline{BC}$. By Euclid V, t_1 and t_2 intersect at a distinct point, D, which is equidistant from A, B, and C by Theorem

2.17. Then a circle O with center D and radius $m\overline{DC}$ passes through A, B, and C. Since the center must lie on the intersection of the perpendicular bisectors of the chords $\overline{AB}$ and $\overline{AC}$, the center D is uniquely determined, hence O is uniquely determined. ∎

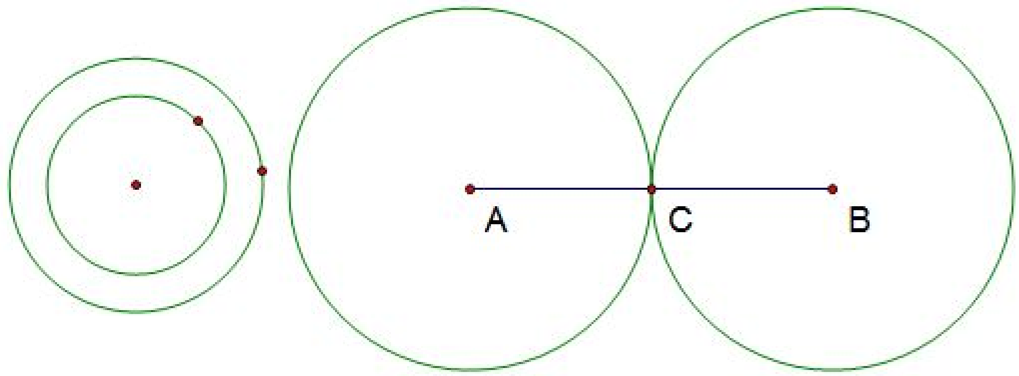
Theorem 4.2 *Given a line l, and two points, A and B, on the same side of l, if $\overleftrightarrow{AB}$ is not perpendicular to l then there exists a unique circle passing through A and B whose center lies on l.*

Proof: Given that points A and B are on the same side of a given line l such that $\overleftrightarrow{AB}$ is not perpendicular to l, Euclid V implies that t, the perpendicular bisector of $\overline{AB}$, intersects l at a unique point C. As C is equidistant from A and B, a circle O with center C and radius $m\overline{CA}$ passes through both A and B. As C is unique, O must also be unique. ∎

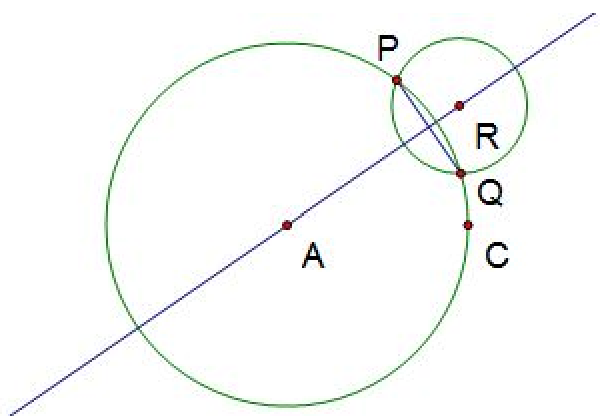


Theorem 4.3 *Two distinct circles may intersect once, twice, or not at all.*

Proof: By Theorem 4.1 if two circles intersect at three or more points, then these two circles are are not distinct. To show that two distinct circles may intersect twice, once, or not at all, we will consider each case individually. The last case is clear; simply consider two circles with the same center and different radii. For the second case, consider points A and B, and the midpoint C of $\overline{AB}$. Letting O_1 and O_2 be circles with center A and B respectively, each with radius $m\overline{AC} = m\overline{BC}$, we have circles that intersect at a single point C. If these two circles were to intersect at another point D, then A and B would each lie on the perpendicular bisector of $\overline{CD}$; such a perpendicular bisector cannot exist by axiom 1.



For the first case, keep A, C, and O_1 as stated, and choose two distinct points of O_1, P and Q. Let R be a point on the perpendicular bisector of $\overline{PQ}$, so that a circle O_3 with center R and radius $m\overline{RP} = m\overline{RQ}$ intersects O_1 at the two points, P and Q. ∎



Definition 4.4 Two circles are called *orthogonal* when the circles intersect, and the lines tangent to the circles at a given point of intersection are perpendicular to each other.

Theorem 4.5 *Given a circle C and two point of that circle P and Q, if P and Q*

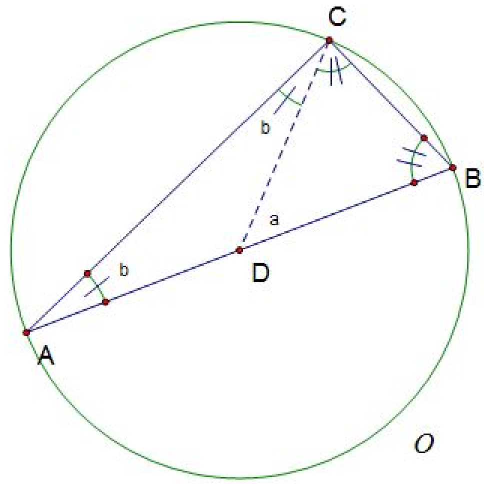
are not diametrically opposed then there exists exactly one circle D passing through P and Q orthogonal to C.

Proof: Given a circle C with center O and two points P and Q, not diametrically opposed on C, take l to be the perpendicular bisector of $\overline{PQ}$. Note that l must pass through O by remark 1. Now, for a circle D to be orthogonal to C, the segments connecting the center of D to P (or Q) and the segment connecting O to P (or Q) must together form a right angle. So we'll take t to be tangent to C at P. Because $\overline{PQ}$ is not a diameter of C, t will intersect l at some point R. Then defining D to be the circle centered at R with radius $m\overline{RP} = m\overline{RQ}$, we have a circle orthogonal to C. Because both t and l are unique, D is unique; no circle distinct from D may pass through P and Q and be orthogonal to C. ∎

Definition 4.6 Given a triangle T, if a circle C contains the three vertices of T, then C is said to *circumscribe* T, and T is *inscribed* in C.

Theorem 4.7 *Given a triangle ΔABC, if a circle O circumscribes this triangle such that $\overline{AB}$ is a diameter of O, then ΔABC is a right triangle, and $\angle ACB$ is the required right angle.*

Proof: Let ΔABC and circle O be as stated, with $\overline{AB}$ a diameter of O and C a point on O. Letting D be the center of O, $m\angle DAC = m\angle DCA$ and $m\angle DBC = m\angle DCB$ since ΔDAC and ΔDBC are isosceles triangles. Then we have that $2m\angle DAC + 2m\angle DBC = 180°$, or $m\angle BAC + m\angle ABC = 90°$. Thus, $\angle ACB$ is a right angle. ∎



Remark 2 *If ΔDCB is an isosceles triangle, and O is a circle with center D and radius $r = m\overline{DB} = m\overline{DC}$, then there is a point A on O such that $A - D - C$; so $\overline{AC}$ is a diameter of O, and $m\angle CDB = 2m\angle CAB$. Referencing the diagram above, Notice that $\angle CDB$ is exterior to ΔADC, so $b = 2a$.*

Theorem 4.8 (Inscribed Angle Theorem) *Given a circle γ with center O and distinct points A, B, and C of γ, $m\angle COB = 2m\angle CAB$.*

Proof: Let γ, A, B, C, and O be as stated; if any side of δABC is already a diameter then we're done, so assume that no side of this triangle is a diameter. Let D be a point of γ such that $\overline{AD}$ is a diameter of γ. Then by the previous theorem

$$\begin{aligned} m\angle BAD &= \frac{1}{2}m\angle BOD \\ m\angle CAD &= \frac{1}{2}m\angle COD \end{aligned}$$

If C is interior to $\angle BAD$, then

$$\begin{aligned} m\angle CAB &= m\angle BAD - m\angle CAD \\ &= \frac{1}{2}(m\angle BOD - m\angle COD) \\ &= \frac{1}{2}m\angle BOC \end{aligned}$$

By symmetric argument if B is interior to $\angle CAD$ then the desired conclusion holds. If D is interior to $\angle CAB$, then

$$\begin{aligned} m\angle CAB &= m\angle CAD + m\angle BAD \\ &= \frac{1}{2}(m\angle COD + m\angle BOD) \\ &= \frac{1}{2}m\angle COB. \end{aligned}$$

∎

4.2 Inversion, Dilation, and Reflection

Though we'll look at dilation and reflection in this section, inversion about a circle is our prime concern. An inversion about a circle C is a type of mapping from the punctured plane onto itself, where points outside C are sent to the interior of C, and points interior to C are mapped outside C, and the only points fixed by this inversion are those points on C.

Definition 4.9 Given a circle C with center O and any point P distinct from O, the *inversion* about C, denoted I_C, gives us $I_C(P) = P'$ where $|OP||OP'| = r_C^2$ and O, P, and P' are collinear (O not between P and P'), and r_C is the radius of C.

Definition 4.10 Let O be a point and k a positive number. The *dilation* with center O and ratio k is the transformation of the Euclidean plane that fixes O and maps a point $P \neq O$ onto the unique point P^* on $\overrightarrow{OP}$ such that $|OP^*| = k|OP|$.

Remark 3 *For each point O and real number $k > 0$, $D_{k,O}$ is a bijection with inverse $D_{k,O}^{-1} = D_{\frac{1}{k},O}$.*

Definition 4.11 Given a line l, *reflection* about a line, denoted R_l, is a bijection from the plane onto itself where, for any point P,

1. if P is not on l and O is the point on l such that $\overleftrightarrow{OP}$ is perpendicular to l, then $R_l(P) = P'$ iff $|OP| = |OP'|$ and $P' - O - P$, or

2. iff P is on l, then $R_l(P) = P$.

Theorem 4.12 *Given a circle C, the inversion I_C, and a point P not the center of C,*

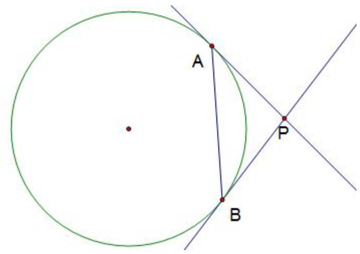
1. *if P is interior to C, then $I_C(P)$ lies exterior to C,*

2. *if P is exterior to C, then $I_C(P)$ lies interior to C, and*

3. *if P lies on C, then $I_C(P) = P$.*

Proof: Each of these should be clear. Case 1: Suppose instead that P' is not exterior to C. Then $|OP'| \leq r_C$, and therefore, $|OP||OP'| < r_C^2$, where r_C is the radius of C. Case 2: Suppose that P' is not interior to C. Then $|OP'| \geq r_C$, and $|OP||OP'| > r_C^2$. Case 3: Suppose that P' is distinct from P, and is therefore either interior or exterior to C. Then $|OP'| \neq r_C$, and therefore $|OP||OP'| \neq r_C^2$. ∎

Definition 4.13 Given a circle γ and a chord $\overline{AB}$ of γ such that $\overline{AB}$ is not a diameter of γ, then the lines tangent to γ at A and B intersect at a point P. This point P is called the *pole* of $\overline{AB}$.

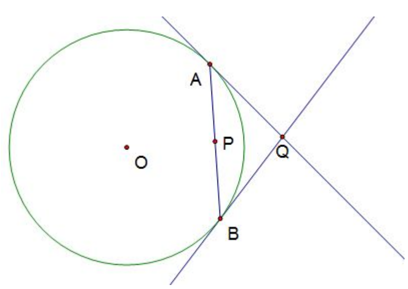


Theorem 4.14 *Given a circle γ with center O and radius r, and a point P distinct from O interior to γ, if $\overline{AB}$ is the chord of γ perpendicular to $\overleftrightarrow{OP}$ at P, then the pole of $\overline{AB}$ is P', the inverse of P through γ.*

Proof: Take the circle γ, point P, and chord of γ, $\overline{AB}$, to be as stated. Then take Q to be the pole of $\overline{AB}$ with respect to γ. Then we have right triangles ΔOPA and ΔOAQ. Since each of these triangles are right triangles, and they share $\angle AOQ$, $\Delta OPA \sim \Delta AOQ$. Then

$$\begin{aligned} \frac{|OP|}{|OA|} &= \frac{|OA|}{|OQ|} \\ \Rightarrow |OP||OQ| &= |OA|^2 \\ &= r^2 \end{aligned}$$

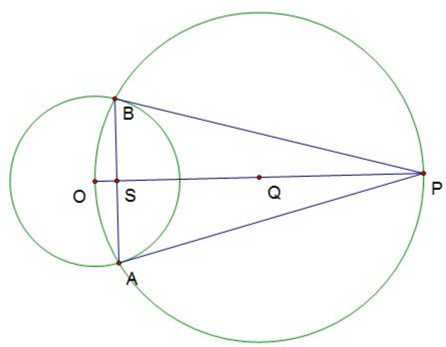
Therefore, the pole of $\overline{AB}$ with respect to γ is the inverse of P through γ. ∎



Remark 4 *Given a circle γ with center O and a point P interior to γ distinct from O, if $\overline{AB}$ is the chord of γ perpendicular to $\overline{OP}$ at P and P' is the pole of P with respect to γ, then $\Delta P'AB$ is an isosceles triangle with $\angle P'AB \cong \angle P'BA$.*

Theorem 4.15 *Suppose a point P is outside a circle γ with center O. Let Q be the midpoint of $\overline{OP}$, and take δ to be the circle with center Q and diameter $\overline{OP}$. Then δ and γ intersect at two points, A and B, $\overleftrightarrow{AP}$ and $\overleftrightarrow{BP}$ are tangent to γ, and the inverse of P is the point at which $\overline{AB}$ intersects $\overline{OP}$.*

Proof: Since δ contains points both inside and outside γ, it must intersect γ at two points A and B. Also, because $\overline{OP}$ is a diameter of δ, for any point R on δ $\angle PRO$ will be a right angle, so $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangent to γ. Now, letting S be the point at which $\overline{AB}$ intersects $\overline{PO}$, by construction P is the pole of S, so by the last theorem P is the inverse of S. Since composition of inversion through a given circle gives the identity function on the punctured plane, S is also the inverse of P. ∎



Theorem 4.16 *Given a circle C with center ω and radius r, the inverse of a point $z \in \mathbb{C} \setminus \{\omega\}$, through C is given by $I_C(z) = \frac{r^2}{\bar{z}-\bar{\omega}} + \omega = \frac{\omega\bar{z}+(r^2-|\omega|^2)}{\bar{z}-\bar{\omega}}$.*

Proof: Begin by noting that the inverse of a point z through the unit circle centered on at the origin is $\frac{1}{\bar{z}}$; $|z||\frac{1}{\bar{z}}| = 1$ is clear, and since $\frac{1}{\bar{z}} = \frac{z}{|z|^2}$, we have that z, $\frac{1}{\bar{z}}$, and the origin are collinear. Not only that, but because $|z|^2 > 0$ the origin cannot lie between z and $\frac{1}{\bar{z}}$. We will use this below, taking the funciton I_1 to be inversion through the unit circle centered at the origin, so that $I_1(z) = \frac{1}{\bar{z}}$.

Now, given any circle C with center ω and radius r, we may find the inverse of a point $z \neq \omega$ through dilations and translations. Using the function $T_\omega(z) = z - \omega$, an example of a translation, we can move our circle C so that it is centered at the origin. Then, with function $D_r(z) = \frac{z}{r}$, we transform the circle C into the unit circle. The function I_1 then gives the inverse of the point to which our original point, z, is sent by T_ω and D_r. Then using the inverse functions $D_r^{-1}(z) = rz$ and $T_\omega^{-1}(z) = z+\omega$, we rescale and translate the unit circle back to C. Defining a function F to be the composition of these functions in the order presented, we have

$$F(z) = (T_\omega^{-1} \circ D_r^{-1} \circ I_1 \circ D_r \circ T_\omega)(z) = \frac{r^2}{\bar{z} - \bar{\omega}} + \omega.$$

In order to check that this meets the requirements for inversion through C, first note that

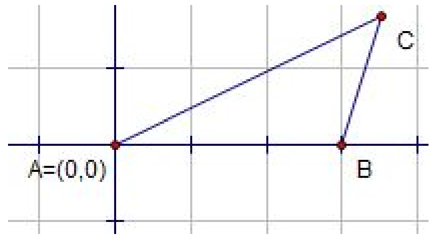
$$\begin{aligned} |z-\omega||F(z)-\omega| &= \sqrt{(z-\omega)(\bar{z}-\bar{\omega})}\sqrt{\left(\frac{r^2}{\bar{z}-\bar{\omega}} + \omega - \omega\right)\left(\frac{r^2}{z-\omega} + \bar{\omega} - \bar{\omega}\right)} \\ &= \sqrt{(z-\omega)(\bar{z}-\bar{\omega})}\sqrt{\frac{r^4}{(\bar{z}-\bar{\omega})(z-\omega)}} \\ &= r^2. \end{aligned}$$

We still need that $F(z)$, z, and ω are collinear such that ω is not between z and $F(z)$. However, the functions T_ω and D_r and their inverses preserve such relations, and because for any $z' \in \mathbb{C} \setminus \{0\}$, the origin, z' and $\frac{1}{z'}$ are collinear such that the origin is not between z' and $\frac{1}{z'}$, the desired relationship holds for z, $F(z)$, and ω. ∎

Remark 5 *Given a circle C, $I_C \circ I_C$ is the identity function on the punctured plane, so I_C is its own inverse.*

Theorem 4.17 *Given a triangle ΔABC with $m\angle CAB = \theta$, $\cos\theta = \frac{|CB|^2-|AC|^2-|AB|^2}{-2|AB||AC|}$. This is known as the Law of Cosines.*

Proof: Given a triangle ΔABC, coordinatize the plane such that $A = (0,0)$, $B = (|AB|, 0)$, and $C = (|AC|\cos\theta, |AC|\sin\theta)$.



Then

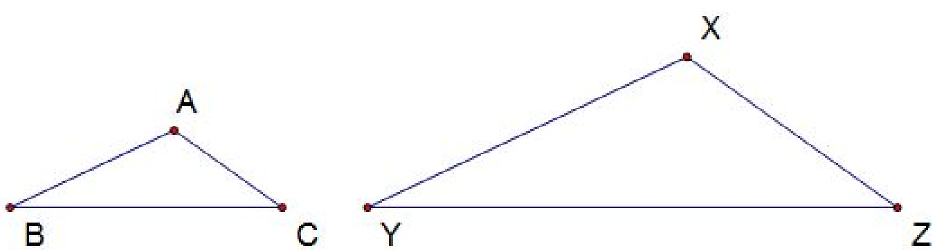
$$|CB| = \sqrt{(|AC|\cos\theta - |AB|)^2 + (|AC|\sin\theta - 0)^2}.$$

Solving this equation for $\cos\theta$, we have

$$\begin{aligned} |CB|^2 &= |AC|^2\cos^2\theta - 2|AC||AB|\cos\theta + |AB|^2 + |AC|^2\sin^2\theta \\ &= |AC|^2 + |AB|^2 - 2|AB||AC|\cos\theta \\ \Rightarrow \cos\theta &= \frac{|CB|^2 - |AB|^2 - |AC|^2}{-2|AC||AB|}. \end{aligned}$$

∎

Theorem 4.18 *Given triangles ΔABC and ΔXYZ, if $\frac{|AB|}{|XY|} = \frac{|BC|}{|YZ|} = \frac{|CA|}{|ZX|}$, then $\Delta ABC \sim \Delta XYZ$.*



Proof: We want to show that $\gamma_1 = m\angle ABC = m\angle XYZ = \gamma_2$. At least two angles of each triangle are acute, so assume that γ_1 and γ_2 are acute. Then using the law of cosines, we have

$$\begin{aligned}
\cos\gamma_1 &= \frac{|CA|^2 - |AB|^2 - |BC|^2}{-2|AB||BC|} \\
&= -\frac{1}{2}\left(\frac{|CA||CA|}{|AB||BC|} - \frac{|AB||AB|}{|AB||BC|} - \frac{|BC||BC|}{|AB||BC|}\right) \\
&= -\frac{1}{2}\left(\frac{|ZX||ZX|}{|XY||YZ|} - \frac{|XY||XY|}{|XY||YZ|} - \frac{|YZ||YZ|}{|XY||YZ|}\right) \\
&= \frac{|ZX|^2 - |XY|^2 - |YZ|^2}{-2|XY||YZ|} \\
&= \cos\gamma_2.
\end{aligned}$$

So $\gamma_1 = \gamma_2$. Letting α_1, α_2, β_1, and β_2 denote the measure of angles $\angle BCA$, $\angle YZX$, $\angle CAB$, and $\angle ZXY$ respectively, the same argument gives us $\cos\alpha_1 = \cos\alpha_2$ and $\cos\beta_1 = \cos\beta_2$.

Assuming that $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$, it must then be that $\alpha_1 = 180 - \alpha_2$ and $\beta_1 = 180 - \beta_2$. Then

$$\begin{aligned}
\alpha_1 + \beta_1 &= 360 - \alpha_2 - \beta_2 \\
&= 180 + \gamma_2 \\
&= 180 + \gamma_1.
\end{aligned}$$

However, the sum of the measure of two angles of any triangle may not be greater than (or equal to) $180°$, so the assumption that $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$ is flawed. Therefore, at least one pair of corresponding angles are congruent, which then implies that all three pairs of angles are congruent. ∎

Theorem 4.19 *Given two triangles ΔABC and ΔXYZ, if two sides of ΔABC are proportional to two sides of ΔXYZ, and if the measure of the included angle of the first pair of sides is equal to the measure of the included angle of the second pair, then $\Delta ABC \sim \Delta XYZ$.*

Proof: Given triangles ΔABC and ΔXYZ such that $|AB|/|BC| = |XY|/|YZ|$ and $m\angle ABC = m\angle XYZ$, let γ denote this angle measure. Then

$$\begin{aligned}\cos\gamma &= \frac{|ZX|^2 - |YX|^2 - |YZ|^2}{-2|YX||YZ|} \\ &= \frac{|CA|^2 - |AB|^2 - |BC|^2}{-2|AB||BC|}.\end{aligned}$$

This leads us to

$$\begin{aligned}\frac{|ZX|^2}{|YX||YZ|} - \frac{|YX|}{|YZ|} - \frac{|YZ|}{|YX|} &= \frac{|CA|^2}{|AB||BC|} - \frac{|AB|}{|BC|} - \frac{|BC|}{|AB|} \\ \Rightarrow \qquad \frac{|ZX|^2}{|CA|^2} &= \frac{|XY||YZ|}{|AB||BC|} = \frac{|XY|^2}{|AB|^2}\end{aligned}$$

Thus, by the previous theorem, $\Delta ABC \sim \Delta XYZ$. ∎

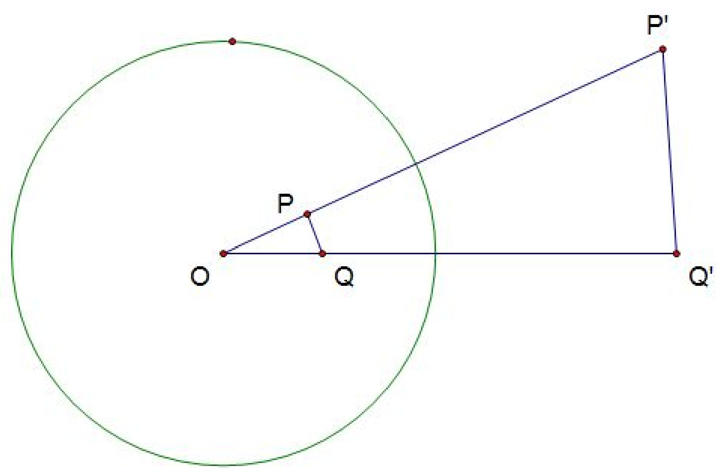
Theorem 4.20 *Given a circle C with center O, and points P and Q, if $I_C(P) = P'$ and $I_C(Q) = Q'$, then $\Delta OPQ \sim \Delta OQ'P'$.*

Proof: Let C, O, P, Q, P', and Q' exist as stated, and let r be the radius of C. Then

$$|OP||OP'| = r^2 = |OQ||OQ'|,$$

or $|OP|/|OQ| = |OQ'|/|OP'|$. Then by the previous theorem, $\Delta OPQ \sim \Delta OQ'P$.

∎



Theorem 4.21 *Given a circle C, and four points A, B, P, and Q, each distinct from the center of C, inversion in C preserves the ratio $\frac{|PA||BQ|}{|PB||AQ|}$; we'll later refer to this as the cross-ratio.*

Proof: Given points A, B, P, and Q, and a circle C, assume that none of the given points are the center of C. We have from inversion through C, $I_C(A) = A'$, $I_C(B) = B'$, $I_C(P) = P'$, and $I_C(Q) = Q'$. Letting r be the radius of C and O it's center, we have by how inversion is defined that

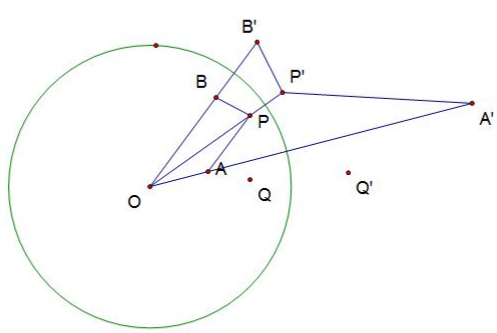
$$|OA||OA'| = |OB||OB'| = |OP||OP'| = |OQ||OQ'| = r^2.$$

By Theorem 4.20, $\Delta OPA \sim \Delta OA'P'$ and $\Delta OPB \sim \Delta OB'P'$. Then $\frac{|PA|}{|P'A'|} = \frac{|OP|}{|OA'|}$ and $\frac{|PB|}{|P'B'|} = \frac{|OP|}{|OB'|}$, which gives

$$\begin{aligned} \frac{|PA|}{|PB|} &= \frac{|P'A'||OP|}{|OA'|}\frac{|OB'|}{|P'B'||OP|} \\ &= \frac{|P'A'||OB'|}{|P'B'||OA'|} \end{aligned}$$

Similarly, $\frac{|BQ|}{|AQ|} = \frac{|Q'B'||OA'|}{|Q'A'||OB'|}$. Thus, we have the desired result, $\frac{|PA||QB|}{|PB||QA|} = \frac{|P'A'||Q'B'|}{|P'B'||Q'A'|}$.

Now, we had assumed that none of A, B, P, and Q were the center of C; this is because inversion through a circle is a mapping from the punctured plane to itself, where the center of the circle of inversion is the missing point. The case where P is the center of the circle is a case we'll visit later. ∎



Theorem 4.22 *Given a circle C with center O,*

1. *I_C sends circles not passing through O to circles not passing through O,*

2. *if δ is a circle passing through O, then I_C sends δ to a line not passing through O which is parallel to the line tangent to δ at O,*

3. *I_C sends lines not passing through O to circles passing through O,*

4. *I_C sends lines passing through O to themselves, and*

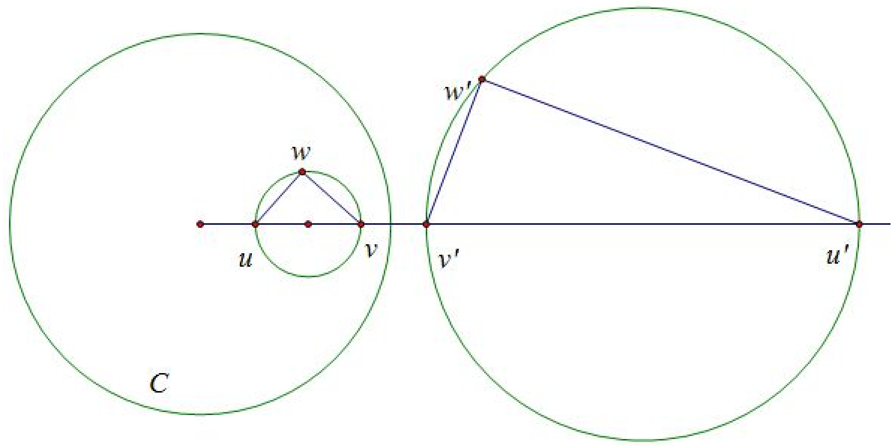
5. *I_C sends circles orthogonal to C to themselves.*

Proof: 1. Let C and γ be circles, with O the center of C and O not on γ. Also, let $\overline{uv}$ be a diameter of γ such that O lies on $\overleftrightarrow{uv}$. If w is any point of γ distinct from u and v, then Δuvw is a right triangle with right angle $\angle uwv$. Let u', v', and w' be the images of u, v, and w through inversion about C. Then $\Delta Owu \sim \Delta Ou'w'$ and $\Delta Owv \sim \Delta Ov'w'$. This similarity gives us congruent angles $\angle Owu \cong \angle Ou'w'$ and $\angle Owv \cong \angle Ov'w'$. However, we know that $m\angle Owv - m\angle Owu = 90$. using the given congruences, this equation gives us

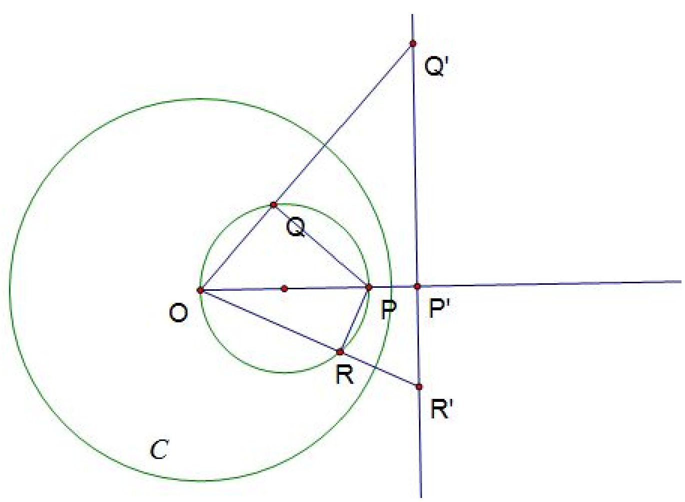
$$\begin{aligned} m\angle Ov'w' - m\angle Ou'w' &= 90 \\ 180 - m\angle w'v'u' - m\angle Ou'w' &= 90 \\ 90 = m\angle w'v'u' + m\angle Ou'w' \end{aligned}$$

Therefore $\angle w'u'v'$ and $\angle w'v'u'$ are complementary, forcing $\angle v'w'u'$ to be a right angle.

This holds for any $w \in \gamma$ distinct from u and v, and therefore any $w' \in I_C(\gamma)$. Therefore, $I_C(\gamma)$ is a circle with diameter $\overline{u'v'}$.



2. Consider the same circle C, and let δ be a circle such that O lies on δ. Letting $\overline{OP}$ be a diameter of δ and Q a point on δ distinct from O and P, we have a right triangle ΔOPQ. Now, $\Delta OPQ \sim \Delta OQ'P'$, so $\angle OP'Q'$ is also a right angle. Letting R be yet another point of δ distinct from O, P, and Q, we also have a right angle $\angle OP'R'$. Since $\overleftrightarrow{P'Q'}$ and $\overleftrightarrow{P'R'}$ are perpendicular to $\overleftrightarrow{OP'}$ at P', these two lines must not be distinct. Therefore, $I_C(\delta)$ is a line perpendicular to $\overleftrightarrow{OP}$ at P'.



Furthermore, since $\overline{OP}$ is a diameter of δ, if t is the line tangent to δ at O then t is also perpendicular to $\overleftrightarrow{OP}$. Since $\overleftrightarrow{OP}$ is a common perpendicular to t and $\overleftrightarrow{P'Q'}$, the image of δ under I_C, we have that $t \parallel \overleftrightarrow{P'Q'}$.

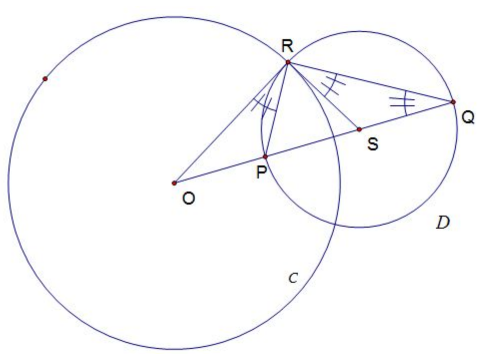
3. Our third case is clear by case two and the preceding theorem.

4. Let l be a line passing through O. Then for any P on l distinct from O, $I_C(P)$ is a point of $\overleftrightarrow{OP}$, but $l = \overleftrightarrow{OP}$. Thus, the image of l under I_C (minus O) is l (minus O).

5. Finally, assume that D is a circle orthogonal to C, and let $\overline{PQ}$ be a diameter of D such that $\overleftrightarrow{PQ}$ passes through O, O the center of C. Let R be a point of intersection between C and D, so that $\angle PRQ$ is a right angle. Letting S be the center of D, we also have that $\angle ORS$ is a right angle. This gives us that $\angle PRS$ is complementary to both $\angle SRQ$ and $\angle ORP$, which gives us $\angle ORP \cong \angle SRQ$. Since ΔSRQ is an isosceles triangle, this last congruence can be extended,

$$\angle ORP \cong \angle SRQ \cong \angle SQR.$$

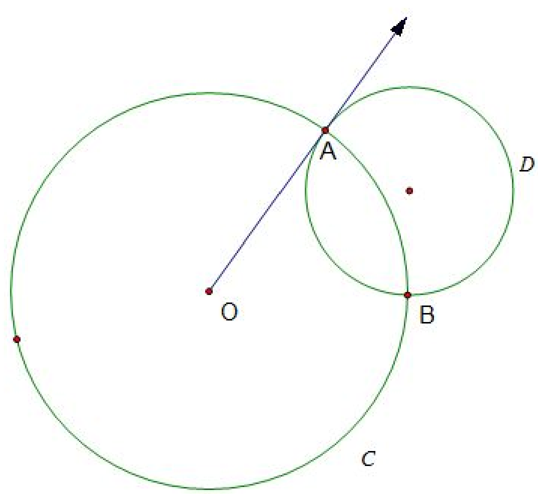
This, along with our assumption that triangles have angle measure 180°, implies that $\angle OPR \cong \angle ORQ$, which in turn implies that $\Delta OPR \sim \Delta ORQ$. With O, P, and Q collinear, and R fixed under inversion through C, it must then be that P and Q are inverses under C. Therefore, the image of d under inversion through C is a circle containing points P, Q, and R; these are points of D, so D is sent to itself under inversion through C. ∎



Theorem 4.23 *Given a circle C, a second circle D is sent to itself under inversion through C if and only if D is orthogonal to C.*

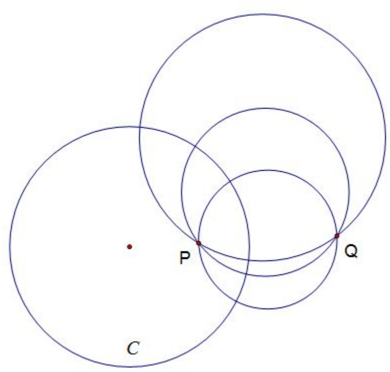
Proof: One direction follows from Theorem 4.22 part 5. For the other direction, consider a circle C with center O, and a circle D is sent to itself under inversion through C. It follows from Theorem 4.22 that some points of D must lie interior to C, and some must lie exterior to C; then C and D must intersect at two points, A and B. Also from Theorem 4.22 (part 4), we know that $\overrightarrow{OA}$ is sent to itself under inversion. Now, either $\overrightarrow{OA}$ is tangent to D or it intersects D at two points, A and some other point A'.

For the sake of contradiction, assume that $\overrightarrow{OA}$ is not tangent to D. Since D and $\overrightarrow{OA}$ are sent to themselves under inversion through C, the intersection $D \cap \overrightarrow{OA} = \{A, A'\}$ is sent to itself, as a set, under inversion through C. However, A is a fixed point under inversion through C, so A' must also be a fixed point. This implies that A' lies on both C and $\overrightarrow{OA}$, which in turn implies that $A = A'$, contradicting that A and A' are distinct. Therefore, if D is sent to itself under inversion through C then $\overrightarrow{OA}$ is tangent to D, so that D is orthogonal to C. ∎



Theorem 4.24 *If P and Q are distinct points and inverses under I_C for some circle C, then any circle passing through P and Q is orthogonal to C.*

Proof: Suppose that $I_C(P) = Q$ for some circle C with center O (Q distinct from P, or equivalently P not on C). Now take D to be a circle passing through P and Q. Since one of P and Q lies inside C and one lies outside C, D must intersect C twice; however, we'll only need to use one point of intersection, so take one point of intersection between D and C to be the point R. Recall that inversion through C sends circles not passing through the center of C to other circles not passing through the center of C. Since P, Q, and O are collinear, D cannot pass through O, so I_C sends D to some circle D'. However, three points determine a unique circle, and since P, Q, and R are points of both D and D', it must be that D and D' are the same circle. So, by the previous theorem, D is orthogonal to C. ∎



Theorem 4.25 *Given a circle γ with center O and radius r, denote dilation with center P and ratio k by $D_{k,P}$. $D_{k,P}$ maps γ to a circle γ^* with center O^* and radius kr, and for any point Q on γ, the line tangent to γ at Q is either parallel to the line tangent to γ^* at Q^* or the two lines are not distinct.*

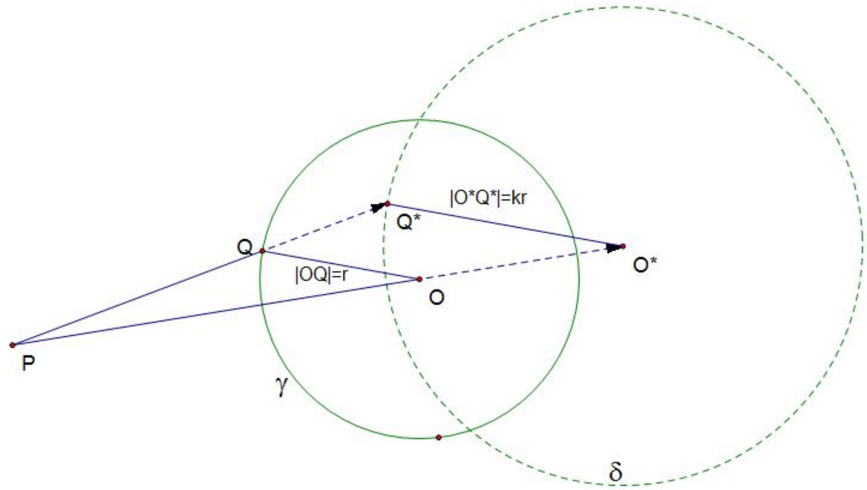
Proof: Consider a circle γ with center O and radius r, and a point P and a positive real number k; we will dilate γ with respect to the point P and ratio k. To begin, note that we can take any point Q, and let $P = p$ and $Q = q$ where p and q are complex numbers. Then consider the ray $z = p + (q - p)t$ with $t \geq 0$. The image of Q under $D_{k,P}$, Q^*, is given by q^* on this ray where $\|q^* - p\| = k\|q - p\|$. Notice that

$$\begin{aligned} k\|q - p\| &= \|k(q - p)\| \\ &= \|p + k(q - p) - p\|, \end{aligned}$$

so $q^* = p + k(q - p)$. Now, if Q is any point on γ then $|QO| = \|q - o\| = r$ where $O = o$, o a complex number. The first we'll show is that $|Q^*O^*| = \|q^* - o^*\| = kr$, so that for any point Q on γ, Q^* lies on a circle with center O^* and radius kr.

$$\begin{aligned} \|q^* - o^*\| &= \|p + k(q - p) - [p + k(o - p)]\| \\ &= \|k(q - o)\| \\ &= k\|q - o\| \\ &= kr. \end{aligned}$$

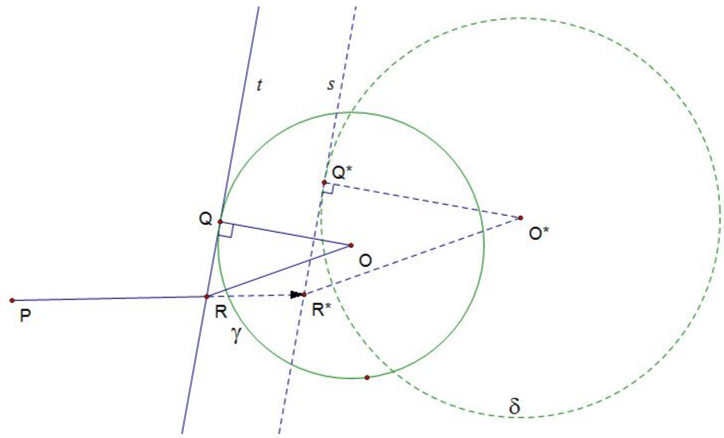
Therefore, $D_{k,P}$ sends a point Q of γ to a point Q^* of a circle δ with center O^* and radius kr. By a symmetric argument, $D_{k,P}^{-1} = D_{\frac{1}{k},P}$ sends each point of δ to a point of γ, so the image of γ under $D_{k,P}$ is δ.



Now, take t to be a line tangent to γ at a point Q, and let R be a point of t distinct from Q. Then $|QR|^2 + |OQ|^2 = |OR|^2$ since ΔOQR is a right triangle with $\angle QRO$ a right angle. We'll show that $\overleftrightarrow{Q^*W^*}$ is a line tangent to δ by showing that $|Q^*R^*|^2 + |O^*Q^*|^2 = |O^*R^*|^2$;

$$\begin{aligned}
|Q^*R^*|^2 + |O^*Q^*|^2 &= \|[p + k(q-p)] - [p + k(r-p)]\|^2 + \|[p + k(o-p)] - [p + k(q-p)]\|^2 \\
&= \|k(q-p) - k(r-p)\|^2 + \|k(o-p) - k(q-p)\|^2 \\
&= k^2(\|q-r\|^2 + \|o-q\|^2) \\
&= k^2\|r-o\|^2 \\
&= \|kr - ko\|^2 \\
&= \|[p + k(r-p)] - [p + k(o-p)]\|^2 \\
&= \|r^* - o^*\|^2 \\
&= |R^*O^*|^2.
\end{aligned}$$

Therefore, $\angle R^*Q^*O^*$ is a right angle for each R on t, so each point of t is sent to a point of s, the line tangent to δ at Q^*. Similarly, $D_{k,P}^{-1}$ sends each point of s to a point of t, so the image of t under $D_{k,P}$ is s, the line tangent to δ at Q^*.



Now, to show that t and s are either parallel or equal, we'll use the equation $z = r + t(q - r)$ for t and the equation $z = r^* + t(q^* - r^*)$ for s. If t and s were to intersect, then there is t' such that

$$\begin{aligned} & r + t'(q - r) = r^* + t'(q^* - r^*) \\ \Rightarrow\quad & r + t'(q - r) = [p + k(r - p)] + t'([p + k(q - p)] - [p + k(r - p)]) \\ \Rightarrow\quad & r + t'(q - r) = (1 - k)p + k[r + t'(q - r)] \\ \Rightarrow\quad & r + t'(q - r) = p \end{aligned} \tag{4.1}$$

However, $r + t(q - r) = z$ is an equation for the line passing through R and Q, so if any solution t' to equation (4.1) exists, then P, Q, and R are collinear; this implies that if such t' exists then $s = t$, since P, Q, and Q^* are collinear and P, R, and R^* are collinear. Therefore, if t and s intersect then they are the same line, and if s and t are distinct lines then they are parallel. ∎

Definition 4.26 Given a circle C with center O and radius r, the *power of a point* P with respect to C is a real number given by $\mathcal{P}(P) = |OP|^2 - r^2$.

Lemma 4.2.1 *If O is a point outside a circle C with center P and radius r, and l is a line through P intersecting C at points Q and R (or at a single point Q if l is tangent to C), then $\mathcal{P}(O) = |OQ||OR|$ (or $\mathcal{P}(O) = |OQ|^2$).*

Proof: This follows from the law of cosines. Consider O, C, P, Q, and R as stated, and the triangles ΔOPQ and ΔOPR. Let $\alpha = m\angle POQ = m\angle POR$. Then the law of cosines gives us $\cos\alpha = \frac{|PQ|^2-|OP|^2-|OQ|^2}{-2|OP||OQ|} = \frac{|PR|^2-|OP|^2-|OR|^2}{-2|OP||OR|}$. Now, consider the following equations/implications:

$$
\begin{aligned}
& \frac{|PQ|^2 - |OP|^2 - |OQ|^2}{-2|OP||OQ|} = \frac{|PR|^2 - |OP|^2 - |OR|^2}{-2|OP||OR|} \\
\Rightarrow\ & (|PQ|^2 - |OP|^2 - |OQ|^2)|OR| = (|PR|^2 - |OP|^2 - |OR|^2)|OQ| \\
\Rightarrow\ & |PQ|^2|OR| - |PR|^2|OQ| + |OP|^2(|OQ| - |OR|) - |OQ||OR|(|OQ| - |OR|) = 0 \\
\Rightarrow\ & r^2|OR| - r^2|OQ| + (|OP|^2 - |OQ||OR|)(|OQ| - |OR|) = 0 \\
\Rightarrow\ & (|OP|^2 - |OQ||OR| - r^2)(|OQ| - |OR|) = 0. \qquad (4.2)
\end{aligned}
$$

By (4.2), either $|OQ| = |OR|$, which implies that $Q = R$, or $|OP|^2 - r^2 = |OQ||OR|$.

If $Q = R$, then ΔOPQ is a right triangle and by the Pythagorean Theorem

$$\mathcal{P}(O) = |OP|^2 - r^2 = |OP|^2 - |PQ|^2 = |OQ|^2$$

as desired. If Q and R are distinct, then $\mathcal{P}(O) = |OP|^2 - r^2 = |OQ||OR|$. ∎

As an alternative proof of Lemma 4.2.1, we may use cyclic quadrilaterals:

Definition 4.27 Given a quadrilateral $\square ABCD$, if A, B, C, and D lie on a circle γ, then $\square ABCD$ is called a *cyclic quadrilateral*.

Theorem 4.28 *If $\square ABCD$ is a cyclic quadrilateral, then opposite angles are supplementary: $m\angle ABC + m\angle CDA = 180°$ and $m\angle BCD + m\angle DAC = 180°$.*

Proof: Suppose that $\square ABCD$ is a cyclic quadrilateral of a circle γ whose center is O. By Theorem 4.8

$$\begin{aligned} m\angle BAC &= \frac{1}{2}m\angle BOC \\ &= m\angle BDC \\ m\angle CAD &= \frac{1}{2}m\angle COD \\ &= m\angle CBD \end{aligned}$$

Now consider the triangles ΔABD and ΔBDC. Because the angle sum of these triangles is $180°$, we have that

$$\begin{aligned} m\angle BCD &= 180 - m\angle BDC - m\angle DBC \\ &= 180 - m\angle CAB - m\angle CAD \\ &= 180 - m\angle BAD, \end{aligned}$$

so $\angle BAD$ and $\angle BCD$ are supplementary; it follows (by symmetric argument or the fact that $\square ABCD$ has angle sum $360°$) that $\angle ABC$ and $\angle CDA$ are also supplementary. ∎

Alternate proof to lemma 4.2.1: If O is a point outside the circle γ, with center is P and radius r, and l and m are lines passing through O, l intersecting γ at points A and B such that $A - B - O$, and m intersecting γ at C and D such that $D - C - O$. Then$\square ABCD$ is a cyclic quadrilateral. By Theorem 4.28 alternate interior angle of $\square ABCD$ are supplementary, so $\angle OBC \cong \angle ODA$ and $\angle OCB \cong \angle OAD$. It follows that $\Delta OBC \sim \Delta OAD$, which implies $\frac{|OB|}{|OD|} = \frac{|OC|}{|OA|}$, or $|OA||OB| = |OC||OD|$. Now, assume that $\overline{CD}$ is in fact a diameter of γ. Then

$$|OA||OB| = |OC||OD| = (|PO| - r)(|PO| + r) = |PO|^2 - r^2 = \mathcal{P}(O).$$

This covers all lines l passing through O and intersecting γ at two distinct points. If l were to be tangent to γ so that l intersects γ at a single point A, then $\angle OAP$ is a right angle. Letting $\overline{CD}$ be a diameter of γ such that $D-C-O$, we have by the Pythagorean Theorem that

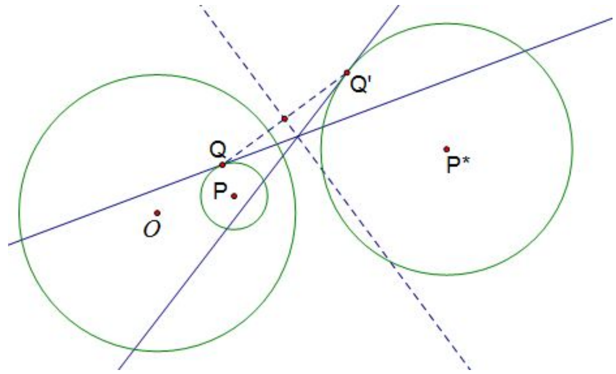
$$\mathcal{P}(O) = |OP|^2 - r^2 = (|OA|^2 + r^2) - r^2 = |OA|^2.$$

Theorem 4.29 *Given a circle γ with radius r and center O, and a circle δ with radius s and center P such that O is outside δ, let p be the power of O with respect to δ. The image of δ under inversions through γ is the circle δ' with radius $\frac{r^2 s}{p}$ and center $P^* = D_{\frac{r^2}{p},O}(P)$. If Q is any point of δ, then the line tangent to δ' at $Q' = I_\gamma(Q)$ is the reflection of the line tangent to δ at Q across the perpendicular bisector of $\overline{QQ'}$.*

Proof: Take γ, δ, and p to be as stated. Then with O outside δ, either $\overrightarrow{OQ}$ intersects δ at a point R distinct from Q, or $\overrightarrow{OQ}$ is tangent to δ, in which case let $Q = R$. Then

$$\frac{|OQ'|}{|OR|} = \frac{|OQ'||OQ|}{|OR||OQ|} = \frac{r^2}{p},$$

so $Q' = D_{\frac{r^2}{p},O}(R)$. This holds for any point Q of δ (and corresponding point R), so inversion through γ sends each point of δ to a point of δ^*; by Theorem 4.25, δ^* is the circle with center P^* and radius $\frac{r^2 s}{p}$. Furthermore, Theorem 4.22 assures us that inversion through γ will send δ to a circle δ', so it must be that $\delta' = \delta^*$.



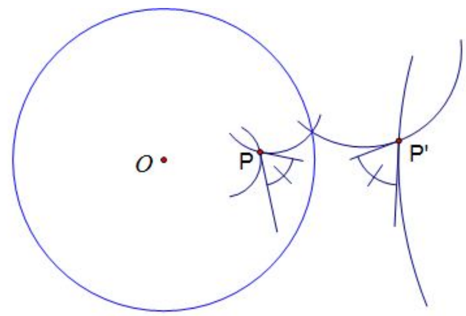
Now, letting t be the line tangent to δ at R, we know that the line t^* tangent to δ^* at $R^* = Q'$ is parallel to t. Furthermore, if $\overline{QR}$ is not a diameter of δ, then the line u tangent to δ at Q will intersect t at some point S, which is the pole of the midpoint M of $\overline{QR}$. So $\angle SRQ \cong \angle SQR$. Since u intersects t, it must also intersect t' at some point T, and we have $\angle SRQ \cong \angle SQR \cong \angle TQ'Q$. This gives us the isosceles triangle $\Delta TQ'Q$, which implies that T is on the perpendicular bisector of $\overline{QQ'}$. Therefore, t' is the reflection of t across the perpendicular bisector of $\overline{QQ'}$. ∎

Definition 4.30 A function ϕ is *conformal* if, for any distinct points A, B, and C in its domain, $m\angle ABC = m\angle\phi(A)\phi(B)\phi(C)$.

Remark 6 *Recall that dilation maps a line t to a line s parallel to t. It follows that dilation is a conformal mapping of the plane onto itself.*

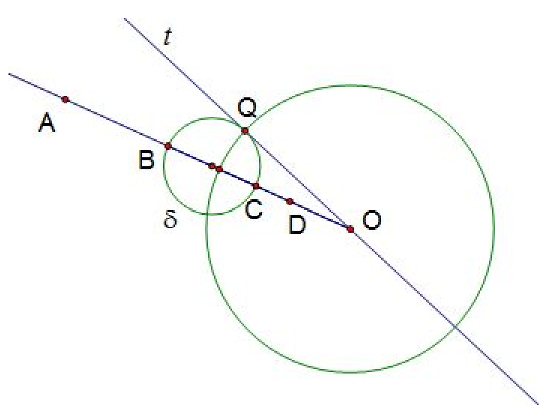
Theorem 4.31 *Inversion is a conformal mapping of the punctured complex plane to itself.*

Proof: Given a circle γ with center O and radius r, let α and β be arcs intersecting at a point P, and let t and s be lines tangent to α and β respectively, both at P. Then the angle formed by the intersection of α and β can be measured with the angle formed by the intersection of t and s. Now, let $I_\gamma(P) = P'$, let α' be the image of α under inversion through γ, and let β' be the image of β. Then the line t' tangent to α' at P' and the line s' tangent to β' at P' are reflections of t and s across the perpendicular bisector of $\overline{PP'}$. Since reflection preserves angles, the angle formed by t' and s is congruent to the angle formed by t and s. Hence, inversion is a conformal mapping. ∎



Theorem 4.32 *Given collinear points A, B, C, D, and O, with $A - B - D - O$, $A - C - D - O$ and $\frac{|AO|}{|BO|} = \frac{|CO|}{|DO|}$, there exists a circle γ with center O such that $I_\gamma(A) = D$, $I_\gamma(B) = C$, and $I_\gamma(\overline{AB}) = \overline{CD}$.*

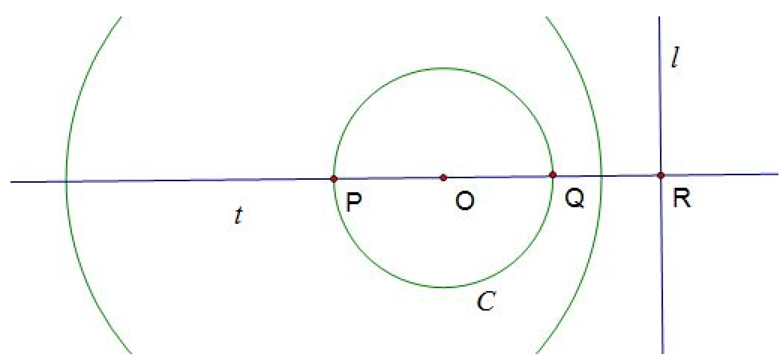
Proof: From our hypothesis, we have that $|AO||DO| = |CO||BO|$. Then let γ be the circle with center O and radius $r = \sqrt{|AO||DO|}$. Then $I_\gamma(A) = D$ and $I_\gamma(B) = C$. ∎



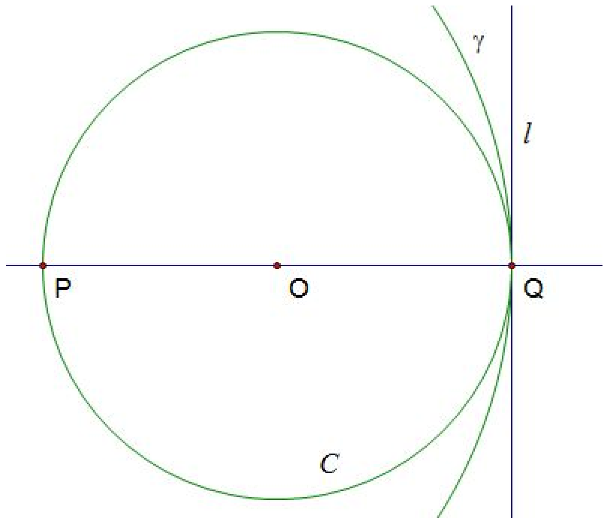
Remark 7 *In order to construct the circle γ used in the proof of Theorem 4.32, first take δ to be the circle for which $\overline{BC}$ is a diameter; the midpoint of $\overline{BC}$, M, will be the center of δ. Letting t be a line tangent to δ at a point Q and passing through O, the radius of γ is then $|OQ|$. This comes from the fact that if two points of a circle (δ) are inverses of one another under inversion through a second circle (γ), then the two circles are orthogonal. Since B and C are points of δ, where B, C, and O are collinear, and $\angle MQO$ is a right angle, the circle γ we've constructed will be orthogonal to δ and inversion through γ will send B to C and visa-versa.*

Theorem 4.33 *Given a circle C with center O and a line l, let t be the line perpendicular to l and passing through O. Then there exists a circle γ such that $I_\gamma(C \setminus \{P\}) = l$, where P is a point of intersection of t with C.*

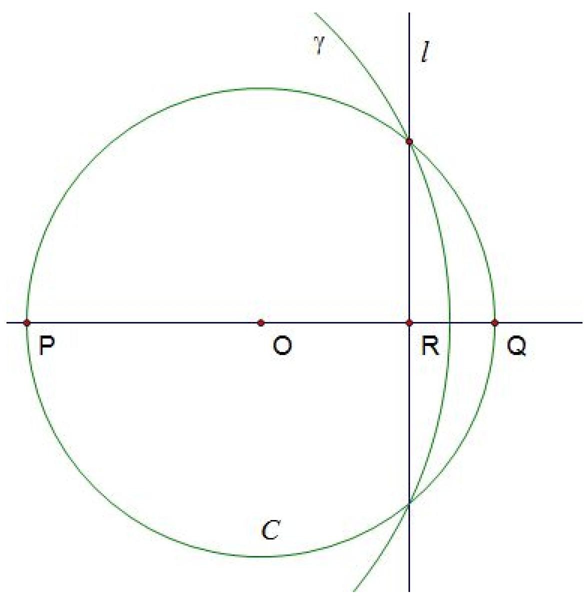
Proof: Let O be the center of our circle C, t be a line perpendicular to l passing through O, and P and Q the points of intersection of t with C. Suppose that t intersects l at R, and consider first the possibility that $P - Q - R$. Then take γ to be the circle with center P and radius $r = \sqrt{|PQ||PR|}$. Then we have that C passes through the center of γ, and that Q and R are inverses with respect to γ since $P - Q - R$ and $|PQ||PR| = r^2$. Then, because $\overleftrightarrow{PR} \perp l$, we have that $I_\gamma(C \setminus \{P\}) = l$.



Notice, however, that if $R = Q$ (so l is tangent to C at Q) then $|PR||PQ| = |PQ|^2 = r^2$, so γ will be the circle with center P and radius $|PQ|$. This is depicted below.

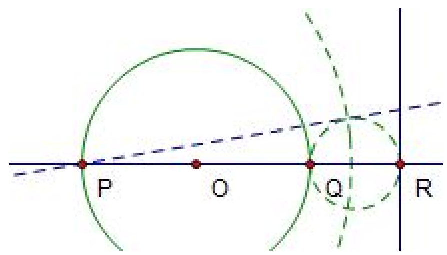


See that if $R = P$, then we may instead consider the circle δ with center Q and radius $|PQ|$; in other words, this does not change our approach, simply the notation. Now, the last case to consider is $P - R - Q$. Then we may again define γ to be the circle with center P and radius $r = \sqrt{|PQ||PR|}$ so that Q and R are inverses with respect to γ, and since $l \perp \overleftrightarrow{PQ}$ we'll have $I_\gamma(C \setminus \{P\}) = l$. ∎



Remark 8 *The construction of the circle γ in the first part of the proof of Theorem*

4.33 is similar to the construction of the circle γ used in the proof of Theorem 4.32, which is explained in remark 7. Take δ to be the circle for which $\overline{QR}$ is a diameter, and construct a line t tangent to δ passing through P. The line t will intersect δ at some point S, and the desired circle γ will be that which has center P and radius $|PS|$.



The construction of γ is a little more transparent for the second and third cases considered in the proof of Theorem 4.33. For the second case, we simply construct the circle with center P and radius PQ. For the third case, notice that because we assume that $P - R - Q$ it must be that l intersects the circle C at some point T; then we may let γ be the circle with center P and radius $|PT|$. These two constructions are illustrated within the proof.

Corollary 4.2.1 *Given any two circles C_1 and C_2, there exists a third circle C such that $I_C(C_1) = C_2$.*

Proof: Given circles C_1 and C_2, let D be a circle such that I_D sends C_2 to some line l and maps C_1 to a circle $I(C_1)$. Then there exists a circle C which maps l to $I_D(C_1)$ by Theorem 4.33.

$$\begin{tikzcd}
C_2 \arrow[r, dashed, "I_{D(C)}"] \arrow[d, "I_D"'] & C_1 \arrow[d, "I_D"] \\
l \arrow[r, "I_C"'] & I_D(C_1)
\end{tikzcd}$$

In the above, $I_{D(C)} = I_D^{-1} I_C I_D$, so that inversion through the circle $D(C)$ maps C_2 to C_1. Note that $D(C)$ may in fact be a line, which we may view as a circle whose center is the point at infinity; in such a case $I_{D(C)}$ is inversion about a circle centered at infinity, which is reflection across a line. ∎

4.3 Möbius Transformations

Definition 4.34 A *Möbius transformation*, or a *linear fractional transformation*, is a (nonconstant) function from the extended complex plane, $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, onto itself $\phi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, of the form

$$\phi(z) = \frac{az + b}{cz + d}.$$

Here, $a, b, c, d \in \mathbb{C}$ with $|ad - bc| \neq 0$. Working in the extended complex plane, $\phi(\infty) = \infty$ if $c = 0$, and $\phi(\infty) = \frac{a}{c}$ and $\phi(\frac{-d}{c}) = \infty$ otherwise.

Theorem 4.35 $\mathbb{M} = \{\phi : \bar{\mathbb{C}} \to \bar{\mathbb{C}} : \phi(z) = \frac{az+b}{cz+d},\ a, b, c, d \in \mathbb{C},\ \text{and}\ ad - bc \neq 0\}$ *is a group under composition.*

Proof: Let $\phi, \psi \in \mathbb{M}$, where $\phi(z) = \frac{az+b}{cz+d}$ and $\psi(z) = \frac{pz+q}{rz+s}$. Then, for closure under composition, consider

$$\begin{aligned} (\phi \circ \psi)(z) &= \frac{a\left(\frac{pz+q}{rz+s}\right) + b}{c\left(\frac{pz+q}{rz+s}\right) + d} \\ &= \frac{apz + aq + brz + bs}{cpz + cq + drz + ds} \\ &= \frac{(ap + br)z + (aq + bs)}{(cp + dr)z + (cq + ds)}. \end{aligned}$$

Now we just need $(ap + br)(cq + ds) - (cp + dr)(aq + bs) \neq 0$. Since,

$$(ap+br)(cq+ds) - (cp+dr)(aq+bs) = apds + brcq - draq - cpbs = (ad-bc)(ps-rq),$$

we have by hypothesis that the two factors on the right side of the above equation are not zero, so the product is not zero and we have closure. An identity element clearly exists, as the identity function $p(z) = z$ is certainly in $\mathbb{M}$. Since composition of functions is associative, the last thing to show is that inverses exist.

Starting with an element of $\mathbb{M}$, $\phi(z) = \frac{az+b}{cz+d}$, consider the associated matrix and it's inverse,

$$\Phi = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \Phi^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

Then, letting

$$\gamma(z) = \frac{\frac{d}{ad-bc}z + \frac{-b}{ad-bc}}{\frac{-c}{ad-bc}z + \frac{a}{ad-bc}} = \frac{dz-b}{-cz+a},$$

we have the composition of ϕ and γ giving

$$\begin{aligned}(\phi \circ \gamma)(z) &= \frac{a\left(\frac{dz-b}{-cz+a}\right) + b}{c\left(\frac{dz-b}{-cz+a}\right) + d} \\ &= \frac{(ad-bc)z + (-ab+ab)}{(cd-cd)z + (-cb+da)} \\ &= z.\end{aligned}$$

So for arbitrary $\phi \in \mathbb{M}$, there exists some $\gamma \in \mathbb{M}$ such that $(\phi \circ \gamma)(z) = z$ (ie: inverses exist).

This gives us that $\mathbb{M}$ is a group under composition. Our next theorem should explain why this approach to finding the inverse of a möbius transformation works.

∎

Theorem 4.36 $\mathbb{M}$ *is isomorphic to* $\frac{\mathrm{GL}_2(\mathbb{C})}{(\mathbb{C}\backslash\{0\})I_2}$, *the general linear group of two by two matrices with complex entries, with complex scalar multiples of the identity matrix factored out.*

Proof: Define $\Sigma : \mathrm{GL}_2(\mathbb{C}) \to \mathbb{M}$ by

$$\Sigma\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{az+b}{cz+d}.$$

Σ is clearly onto, and, letting $\phi(z) = \frac{az+b}{cz+d}$ and $\psi(z) = \frac{pz+q}{rz+s}$, the following shows that Σ is a homomorphism:

$$\begin{aligned}
\Sigma\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix}\right) &= \Sigma\left(\begin{bmatrix} ap+br & aq+bs \\ cp+dr & cr+ds \end{bmatrix}\right) \\
&= \frac{(ap+br)z+(aq+bs)}{(cp+dr)z+(cq+ds)} \\
&= (\phi \circ \psi)(z) \\
&= \Sigma\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \circ \Sigma\left(\begin{bmatrix} p & q \\ r & s \end{bmatrix}\right).
\end{aligned}$$

Since the identity of $\mathbb{M}$ is $\alpha(z) = z$, the kernel of Σ is clearly $(\mathbb{C} \setminus 0)I_2$, the set of complex scalar multiples of the 2×2 identity matrix, so the first isomorphism theorem then gives us the desired result. ∎

Theorem 4.37 *If $\phi \in \mathbb{M}$ then ϕ is the composition of translations, dilations, rotations, and complex inversion.*

Note: complex inversion is the composition of complex conjugation $z \to \bar{z}$, which is simply reflection across the real axis, and inversion through the unit circle S^1, $z \to \frac{1}{\bar{z}}$. So complex inversion is given by $z \to \frac{1}{z}$

Proof: Consider the generic möbius transformation $\phi(z) = \frac{az+b}{cz+d}$. Then

$$\phi(z) = \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} = \frac{a}{c} - \frac{1}{cz+d}\frac{ad-bc}{c}.$$

Now, define function T_α, D_β, R_θ, C, and V as follows:

$$\begin{aligned}
T_\alpha(z) &= \alpha + z && \textit{translation}, \ \alpha \in \mathbb{C} \\
D_\beta(z) &= \beta z && \textit{dilation}, \ \beta \in \mathbb{R} \\
R_\theta(z) &= e^{i\theta} z && \textit{rotation}, \ \theta \in \mathbb{R} \\
C(z) &= \bar{z} && \textit{conjugation, or reflection across the real axis} \\
V(z) &= \frac{1}{\bar{z}} && \textit{inversion through } S^1
\end{aligned}$$

We can then write ϕ in terms of these function. Then, taking $c = r'e^{i\theta'}$ for some r' and θ', and $\frac{ad-bc}{c} = r''e^{i\theta''}$ for some r'' and θ'', we have for any $z \in \mathbb{C}$,

$$\begin{aligned}
z &\xrightarrow{R_{\theta'}} e^{i\theta'} z \\
&\xrightarrow{D_{r'}} r'e^{i\theta'} z = cz \\
&\xrightarrow{T_d} cz + d \\
&\xrightarrow{C \circ V} \frac{1}{cz+d} \\
&\xrightarrow{D_{-r''} \circ R_{\theta''}} -r''e^{i\theta''} \frac{1}{cz+d} \\
&= -\frac{ad-bc}{c} \frac{1}{cz+d} \\
&\xrightarrow{T_{a/c}} \frac{a}{c} - \frac{ad-bc}{c} \frac{1}{cz+d} = \phi(z)
\end{aligned}$$

Therefore, ϕ is the desired composition,

$$\phi(z) = (T_{\frac{a}{c}} \circ D_{-r''} \circ R_{\theta''} \circ C \circ V \circ T_d \circ D_{r'} \circ R_{\theta'})(z).$$

∎

Corollary 4.3.1 *If $\phi \in \mathbb{M}$, then ϕ is conformal.*

Proof: This follows from the previous theorem. Clearly translations, reflections, rotations, and dilations are conformal. We've previously shown that inversions are conformal. Therefore möbius transformations, compositions of such mappings, are also conformal. ∎

Corollary 4.3.2 *An element of* $\mathbb{M}$ *sends a line or circle to another line or circle.*

Proof: This is almost immediate from Theorem 4.37. Möbius transformations are compositions of translations, dilations, rotations, reflection across the real axis, and inversion through S^1; it is clear that translation, dilations, rotations, and reflections send a line or circle to a line or circle. Then we need only consider what the image of a line or circle is under inversion. However, we've already seen in Theorem 4.22 that inversion through a given circle will send a line or circle to another line or circle, so möbius transformations must send a line or circle to another line or circle. ∎

Theorem 4.38 *If an element of* $\mathbb{M}$ *is not the identity function, then it fixes either one point or two of the extended complex plane.*

Proof: Suppose the $\phi \in \mathbb{M}$ is not the identity function. If $\phi(z) = z$, where $z \neq \infty$, then

$$az + b = cz^2 + dz \quad \text{or} \quad cz^z + (d - a)z - b = 0 \tag{4.3}$$

This equation has either one or two distinct solutions if $c \neq 0$. On the other hand, ∞ is a fixed point of ϕ iff $c = 0$, and (4.1) becomes $(d - a)z - b = 0$, which has exactly one solution if $d \neq a$ and none if $d = a$. Thus, a non-trivial element of $\mathbb{M}$ fixes exactly one or exactly two points of the extended complex plane. ∎

Corollary 4.3.3 *Each möbius transformation is uniquely determined by its effect on any three distinct points.*

Proof: Let p, q, and r be any three distinct points of the extended complex plane, and suppose that m_1 and m_2 are möbius transformations such that

$$m_1(p) = m_2(p),\ m_1(q) = m_2(q),\ \text{and}\ m_1(r) = m_2(r).$$

Then $m_1^{-1}m_2$ fixes p, q, and r, which implies that $m_1^{-1}m_2 = I$ is the identity function by Theorem 4.38, which in turn implies that $m_1 = m_2$. ∎

Remark 9 *By Corollary 4.3.3, given any elements p, q, and r of the extended complex plane, there exists a unique möbius transformation N such that $N(p) = 0$, $N(q) = 1$, and $N(r) = \infty$, namely*

$$N(z) = \left(\frac{q-r}{q-p}\right)\left(\frac{z-p}{z-r}\right).$$

This is denoted $[z, p, q, r]$.

Note that the cross-ratio $\frac{|AP||BQ|}{|BP||AQ|}$ is often denoted (AB, PQ), and that if we identify points A, B, P, and Q with elements of the extended complex plane a, b, p, and q respectively, then

$$(AB, PQ) = \frac{|AP||BQ|}{|BP||AQ|} = \frac{\|a-p\|\|b-q\|}{\|a-q\|\|b-p\|} = \|[b, q, a, p]\|.$$

Now, suppose that we have $N(z) = [z, p, q, r]$ for some p, q, and r, and suppose that m is any möbius transformation. Then notice that $N \circ m$ maps $m^{-1}(p)$ to 0, $m^{-1}(q)$ to 1, and $m^{-1}(r)$ to ∞. Then we have that

$$N \circ m(z) = [z, m^{-1}(p), m^{-1}(q), m^{-1}(r)] = [m(z), p, q, r].$$

It follows that

$$[m(z), m(p), m(q), m(r)] = [z, p, q, r]$$

for any p, q, and r in the extended complex plane. This gives us that, if a, b, p, and q are elements of the extended complex plane associated with points A, B, P, and Q, and m is any möbius transformation, then

$$(AB, PQ) = \|[b, q, a, p]\| = \|[m(b), m(q), m(a), m(p)]\|.$$

What we have here a proof of the following theorem.

Theorem 4.39 *Möbius transformations preserve the cross ratio* $\frac{|AP||BQ|}{|BP||AQ|}$.

Proof: We've seen one approach to proving this theorem. Another approach we may take hinges upon Theorem 4.37, which gives us that möbius transformations are compositions of translations, dilations (with center at the origin), rotations, and complex inversion (complex conjugation composed with inversion through S^1); we'll take on each of these functions on their own, letting A, B, P, and Q be points in the complex plane represented by the complex numbers a, b, p, and q respectively. Translation by α, T_α clearly preserves our ratio since translations preserve (euclidean) distance:

$$|T_\alpha(A)T_\alpha(P)| = [(a+\alpha) - (p+\alpha)]\overline{[(a+\alpha) - (p+\alpha)]} = (a-p)\overline{(a-p)} = |AP|.$$

The same holds true for rotations,

$$|R_\theta(A)R_\theta(P)| = (e^{i\theta}a - e^{i\theta}p)\overline{(e^{i\theta}a - e^{i\theta}p)} = e^{i\theta}(a-p)e^{-i\theta}\overline{(a-p)} = (a-p)\overline{(a-p)} = |AP|,$$

and conjugation,

$$|C(A)C(P)| = (\bar{a} - \bar{p})\overline{(\bar{a} - \bar{p})} = (\bar{a} - \bar{p})(a-p) - |AP|.$$

Dilation, on the other hand, clearly does not preserve (euclidean) distance. However, dilations with center at the origin clearly *do* preserve the ratio we're concerned with;

consider a dilation D_k with center at the origin and ratio k,

$$\frac{|D_k(A)D_k(P)||D_k(B)d_k(Q)|}{|D_k(B)D_k(P)||D_k(A)D_k(Q)|} = \frac{k|AP|k|BQ|}{k|BP|k|AQ|} = \frac{|AP||BQ|}{|BP||AQ|}.$$

Lastly we have inversion. By Theorem 4.21 inversion preserves the ratio $\frac{|AP||BQ|}{|BP||AQ|}$, and our proof is complete; another more analytic approach follows.

Let A, B, P, and Q be points in the complex plane. Then for $\phi \in \mathbb{M}$, $\phi(A) = \frac{aA+b}{cA+d}$ and $\phi(P) = \frac{aP+b}{cP+d}$ for some $a, b, c, d \in \mathbb{C}$. Then

$$\begin{aligned}
\frac{|\phi(A)\phi(P)|}{|\phi(B)\phi(P)|} &= \frac{(\phi(A)-\phi(P))(\overline{\phi(A)}-\overline{\phi(P)})}{(\phi(B)-\phi(P))-(\overline{\phi(B)}-\overline{\phi(P)})} \\
&= \frac{\left(\frac{(aA+b)(cP+d)-(cA+d)(aP+b)}{(cA+d)(cP+d)}\right)\overline{\left(\frac{(aA+b)(cP+d)-(cA+d)(aP+b)}{(cA+d)(cP+d)}\right)}}{\left(\frac{(aB+b)(cP+d)-(cB+d)(aP+b)}{(cB+d)(cP+d)}\right)\overline{\left(\frac{(aB+b)(cP+d)-(cB+d)(aP+b)}{(cB+d)(cP+d)}\right)}} \\
&= \frac{\left(\frac{adA+bcP-bcA-daP}{c^2P+cd(A+P)+d^2}\right)\overline{\left(\frac{adA+bcP-bcA-daP}{c^2AP+cd(A+P)+d^2}\right)}}{\left(\frac{adB+bcP-bcB-daP}{c^2P+cd(B+P)+d^2}\right)\overline{\left(\frac{adB+bcP-bcB-daP}{c^2BP+cd(B+P)+d^2}\right)}} \\
&= \frac{\left(\frac{(ad-bc)(A-P)}{(cA+d)(cP+d)}\right)\overline{\left(\frac{(ad-bc)(A-P)}{(cA+d)(cP+d)}\right)}}{\left(\frac{(ad-bc)(B-P)}{(cB+d)(cP+d)}\right)\overline{\left(\frac{(ad-bc)(B-P)}{(cB+d)(cP+d)}\right)}} \\
&= \frac{(A-P)(cB+d)\overline{(A-P)(cB+d)}}{(B-P)(cA+d)\overline{(B-P)(cA+d)}}
\end{aligned}$$

Then for the desired ratio we have

$$\begin{aligned}
\frac{|\phi(A)\phi(P)|}{|\phi(B)\phi(P)|}\frac{|\phi(B)\phi(Q)|}{|\phi(A)\phi(Q)|} &= \frac{(A-P)(cB+d)\overline{(A-P)(cB+d)}}{(B-P)(cA+d)\overline{(B-P)(cA+d)}}\frac{(B-Q)(cA+d)\overline{(B-Q)(cA+d)}}{(A-Q)(cB+d)\overline{(A-Q)(cB+d)}} \\
&= \frac{(A-P)\overline{(A-P)}(B-Q)\overline{(B-Q)}}{(B-P)\overline{(B-P)}(A-Q)\overline{(A-Q)}} \\
&= \frac{|AP||BQ|}{|BP||AQ|}
\end{aligned}$$

∎

Chapter 5

The Upper-Half Plane as A Model

In this chapter we'll see how $\mathbb{H}$ may serve as a model for Hyperbolic Geometry. In order to do this, we will define some basic terminology as it applies to $\mathbb{H}$, and show that it satisfies all of the planar axioms for Neutral Geometry, as well as the Hyperbolic Axiom. I say *planar* because axioms 5 through 8, along with 10, deal with space, and the upper half plane is intended as a representation of two dimensional Hyperbolic Geometry[1]. Now, before we delve into our axioms, we will need to solidify some definitions, review some terminology, and introduce some new notation.

Notation: $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of the complex number z respectively; so for $z = x + iy$, $\Re(z) = x$ and $\Im(z) = y$.

Notation: Let $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ be our half plane.

Notation: Given a familiarity with the Euclidean geometry, we will use **e** to denote an object's existence as Euclidean, and **h** as Hyperbolic.

For example, $\mathbf{e}\overleftrightarrow{AB}$ is a Euclidean line (or **e**-line) passing through points A and B, and $\mathbf{h}\overleftrightarrow{AB}$ the Hyperbolic line through A and B. This will be an important tool,

[1] There is, however, an analogous model for three-dimensional hyperbolic space called the "half space" model.

since our models of Hyperbolic Geometry are embedded within models of Euclidean Geometry, and Euclidean tools will be used to build up, and show the consistency of, our model. One of these may be assumed at times, which should be clear from context, and in such cases one or both of these notations will be discarded.

Notation: Let $L = \{z \in \mathbb{C} : \Im(z) = 0\}$, and note that L will serve as the boundary of $\mathbb{H}$.

Definition 5.1 Points on L are called *ideal points*, and points outside both L and $\mathbb{H}$ are called *ultra-ideal points.* Note that neither ideal nor ultra-ideal points are actually points of our hyperbolic plane.

Now, lines in the upper half plane will appear as one of two familiar Euclidean objects: first, we have **h**-lines represented as the intersection of **e**-lines, which are perpendicular to L, with $\mathbb{H}$, and second we have **h**-lines represented as the intersection of **e**-circles, whose center lies on L, with $\mathbb{H}$.

We will introduce more notation and definitions along the way, but we have what we need for now, so let's move on to the first axiom.

Axiom 1: Given any two distinct points, there is exactly one line that contains them.

This axiom's standing in the model follows from some Euclidean results. First, let A and B be distinct points of $\mathbb{H}$. If A and B lie on an **e**-line l perpendicular to L at a point C, then $\mathbf{h}\overleftrightarrow{AB} = l \cap \mathbb{H}$ is as unique in $\mathbb{H}$ as l is in $\mathbb{C}$. If the **e**-line through A and B is not perpendicular to L, then by Theorem 4.2 there is exactly one **e**-circle C through A and B whose center is the intersection of L with the perpendicular bisector of $\overline{AB}$, and we will take $\mathbf{h}\overleftrightarrow{AB} = C \cap \mathbb{H}$, giving us a unique line passing through A and B. Thus, our model satisfies our first axiom.

Axiom 2: To every pair of distinct points, there corresponds a positive number. This number is called the distance between the two points.

To begin discussing distance, we'll introduce the Poincaré metric:

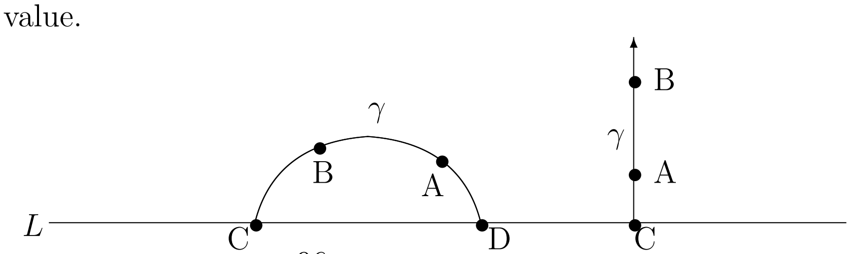
$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2).$$

This discussion takes on two sides: first we have the possibility that two points determine an **h**-line appearing as an **e**-ray, and second we have the appearance of the **h**-line determined by two points as an **e**-semicircle. In either case take γ to be the line we're interested in.

Considering the first case, take $A, B \in \mathbb{H}$ such that $A = x_0 + ia$ and $B = x_0 + ib$, and letting $b > a$. Then we will define the hyperbolic distance (in the half-plane) between these points, $d_{\mathbb{H}}(A, B)$, by the line integral

$$\begin{aligned} d_{\mathbb{H}}(A,B) &= \int_\gamma |ds| \\ &= \int_a^b \frac{\sqrt{(x')^2+1}}{y} dy \\ &= \int_a^b \frac{1}{y} dy \\ &= \ln(y)|_a^b \\ &= \ln\left(\frac{b}{a}\right) \end{aligned} \tag{5.1}$$

Now, if C is the ideal point of this **h**-line $\overleftrightarrow{AB}$, then we have that $d_{\mathbb{H}}(A, B) = \ln(|CB|/|CA|)$ (using the notation $|AC|$ to represent the Euclidean distance from A to C). Note that, by the ordering $b > a$, we have that $|BC|/|AC| > 1$, giving our distance a positive value.



For our second case, we'll have two ideal points for $\mathbf{h}\overleftrightarrow{AB}$, C and D, and another ideal point, $O = c + i0$, as the center of the **e**-semicircle used to define our **h**-line. Assume that $C - B - A - D$ in the hyperbolic sense, and label $m_e\angle DOA = \alpha$ and $m_e\angle DOB = \beta$, allowing $\alpha < \beta$. Then, by representing $\mathbf{h}\overleftrightarrow{AB}$ as

$$\{z : z = c + r\cos\theta + i\sin\theta, 0 < \theta < 180°\}$$

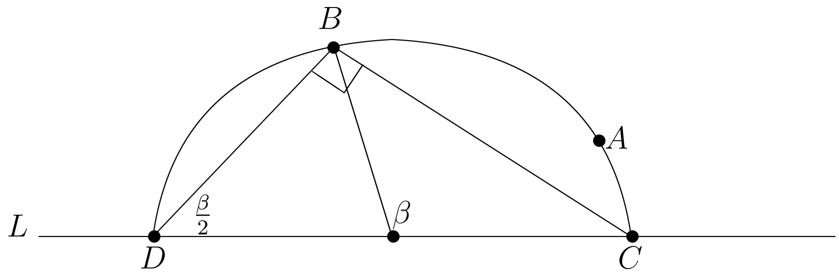
where $r = |OA|$ is the radius of the **e**-semicircle, we have $dx = -r\sin\theta d\theta$, and $dy = r\cos\theta d\theta$, and this allows us to use define the **h**-distance in the half-plane between A and B as follows:

$$\begin{aligned} d_{\mathbb{H}}(A,B) &= \int_\gamma |ds| \\ &= \int_\alpha^\beta \frac{\sqrt{r^2\sin^2\theta + r^2\cos^2\theta}}{r\sin\theta}d\theta \\ &= \int_\alpha^\beta \frac{1}{\sin\theta}d\theta \\ &= \ln\left(\frac{1-\cos\theta}{\sin\theta}\right)\Big|_\alpha^\beta \\ &= \ln\left(\frac{1-(1-2\sin(\theta/2))}{2\sin(\theta/2)\cos(\theta/2)}\right)\Big|_\alpha^\beta \\ &= \ln\left(\tan\left(\frac{\theta}{2}\right)\right)\Big|_\alpha^\beta \\ &= \ln\left(\frac{\tan(\beta/2)}{\tan(\alpha/2)}\right) \end{aligned}$$

At this point, we may use a result regarding circles from Euclidean geometry (illustrated below) to give us that $\tan\frac{\beta}{2} = \frac{|BD|}{|BC|}$, and $\tan\frac{\alpha}{2} = \frac{|AD|}{|AC|}$, so that

$$d_{\mathbb{H}}(A,B) = \ln\left(\frac{|BD|/|BC|}{|AD|/|AC|}\right) = \ln\left(\frac{|BD||AC|}{|AD||BC|}\right).$$

By our ordering, we have that $|BD|/|AD| > 1$ and $|AC|/|BC| > 1$, guaranteeing that our **h**-distance is positive.



Axiom 3: The points of a line can be placed in a correspondence with the real numbers such that

1. To every point of the line there corresponds exactly one real number,
2. To every real number there corresponds exactly one point of the line, and
3. The distance between two distinct points is the absolute value of the difference of the corresponding real numbers.

For this, take l to be an **h**-line, and let A' to be a point on l, assigning to A' the value 0. Assuming that l has only one ideal point, C, take two arbitrary points, P and Q, from l and assign to P and Q the values $\ln(|PC|/|A'C|)$ and $\ln(|QC|/|A'C|)$. If $P - A' - C$ then the value assigned to P, $v_l(P)$, is positive, and the greater the **e**-distance from A' the larger the value assigned to P will be. Similarly, is $A' - P - C$ then $v_l(P)$ will be negative, and as P approaches C, the value assigned to P approaches $-\infty$. More specifically, because we're assigning values using the natural log function, a bijection between $(0, \infty)$ and $\mathbb{R}$, each point will be assigned a unique real number, and every real number will be assigned to exactly one point of l. This takes care of the first and second parts of this axiom. For the third, consider the following:

$$\begin{aligned} |v_l(P) - v_l(Q)| &= |\ln(|PC|/|A'C|) - \ln(|QC|/|A'C|)| \\ &= |\ln(|PC|/|QC|) \\ &= d_{\mathbb{H}}(P,Q). \end{aligned}$$

We've covered axiom 3 for l having only on ideal point, so now assume that l has two ideal points, C and D. Again taking P and Q from l, and assigning values $v_l(P) = \ln\left(\frac{|A'C||PD|}{|PC||A'D|}\right)$ and $v_l(P) = \ln\left(\frac{|A'C||QD|}{|QC||A'D|}\right)$. Again we will have each point of l assigned a unique real number, and each real number assigned to a unique point, and the third part of this axioms holds:

$$\begin{aligned} |v_l(P) - v_l(Q)| &= |\ln\left(\frac{|A'C||PD|}{|PC||A'D|}\right) - \ln\left(\frac{|A'C||QD|}{|QC||A'D|}\right)| \\ &= |\ln\left(\frac{|QC||PD|}{|PC||QD|}\right) \\ &= d_{\mathbb{H}}(P,Q). \end{aligned}$$

Axiom 4: Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

This is essentially covered in the discussion of the previous axiom. However, we said that the ordering of the points would affect the value assigned; this need not occur.

Just as before, given an **h**-line l, choose a point P from l, and assign to it the value zero, $v_l(P) = 0$. Taking another point Q from l, first assume that l has only one ideal point, C. If $Q - P - C$, then $v_l(Q) = \ln(|QC|/|PC|)$ will assign to Q a positive value as discussed earlier. If instead $P - Q - C$, then take $v_l(Q) = \ln(|PC|/|QC|) > 0$.

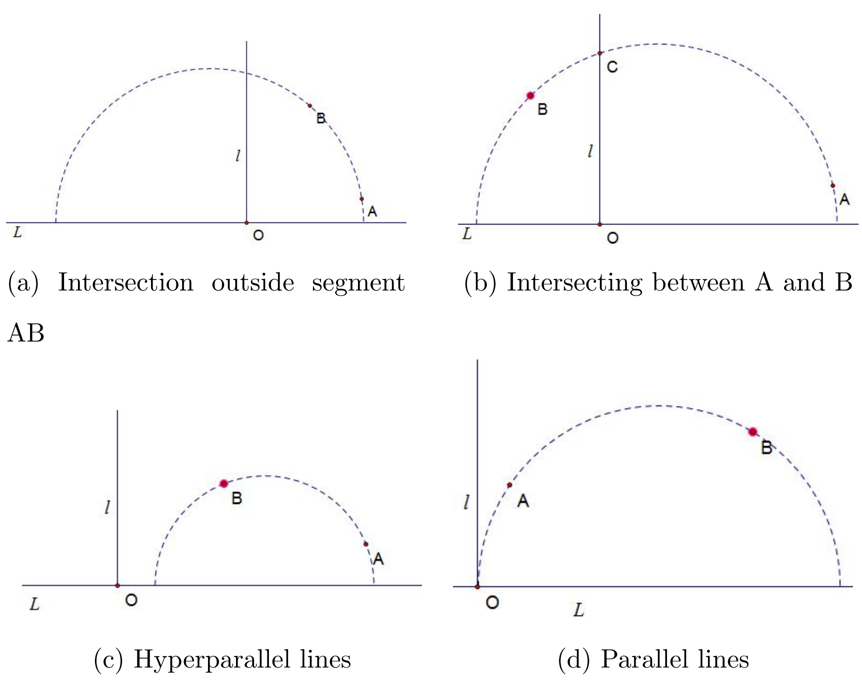
Now, instead assuming that L has two ideal points C and D, if $D - Q - P - C$, then take the same approach to assigning a value to Q: $v_l(Q) = \ln\left(\frac{|PD||QC|}{|QD||PC|}\right) > 0$. If instead $D - P - Q - C$, then assign to Q the value $v_l(Q) = \ln\left(\frac{|PC||QD|}{|QC||PD|}\right) > 0$.

Recall that axioms 5 through 8 deal with 3 dimensions, so we will skip those and move on to the ninth axiom.

Axiom 9: Given a line and a plane containing it, the points of the plane that do not lie on the line form two sets such that

1. each of the sets is convex and
2. if P is in one set and Q is in the other, then segment $\overline{PQ}$ intersects the line.

Given that lines are represented as either **e**-rays or **e**-semicircles with centers on L, that our model satisfies this axiom should be clear. What follows are illustrations of possible situations related to this axiom.



(a) Intersection outside segment AB

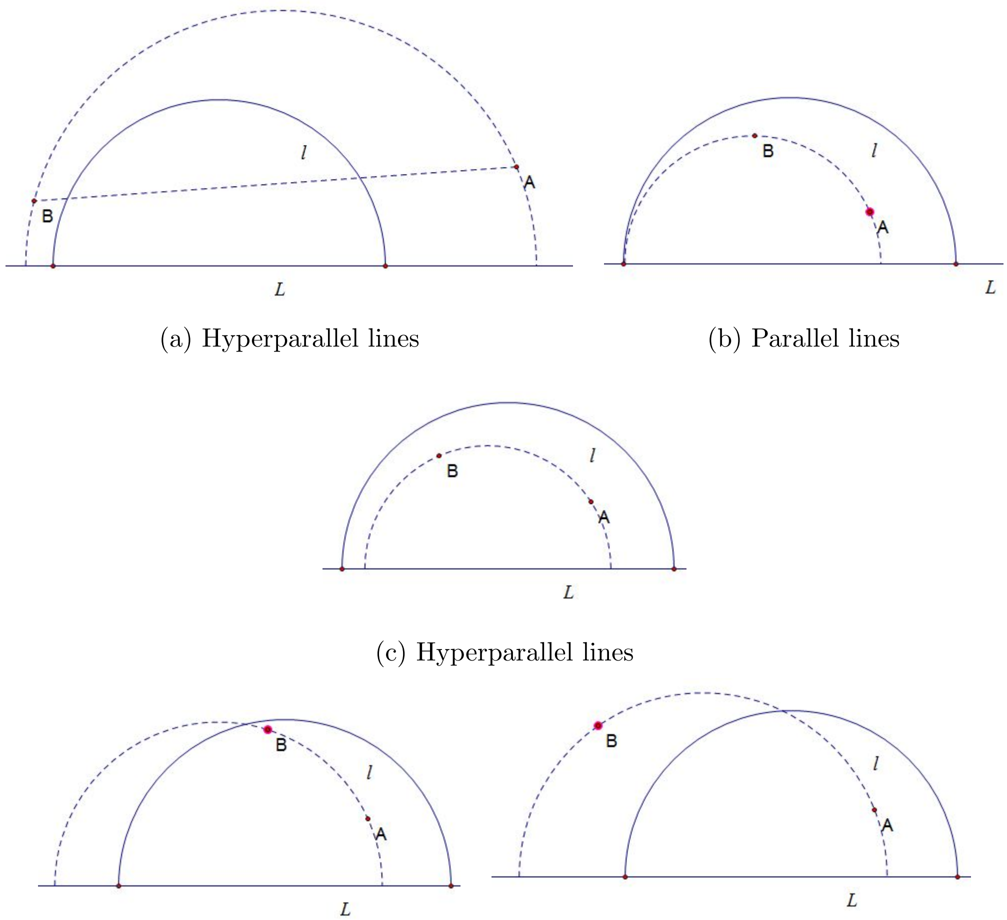
(b) Intersecting between A and B

(c) Hyperparallel lines

(d) Parallel lines

Figure 5.1: Cases with l appearing as an **e**-ray

In (c) and (d) we see a new idea: hyperparallel lines and parallel lines. Though $\overleftrightarrow{AB}$ appear to intersect at O, remember that O is not part of $\mathbb{H}$, l and $\overleftrightarrow{AB}$ merely share an ideal point. In such cases, l and $\overleftrightarrow{AB}$ are said to be parallel or convergent parallels. If $\overleftrightarrow{AB}$ and l do not intersect and do not both converge to an ideal point, then they are said to be hyperparallel.



(a) Hyperparallel lines (b) Parallel lines

(c) Hyperparallel lines

(d) Intersection outside segment AB (e) Intersection at C between A and B

Figure 5.2: Cases with l appearing as an **e**-semicircle

Notice in figure 5.2(a), that the **e**-segment $\overline{AB}$ passes through l, so the sets l separates the plane into are not convex in the Euclidean sense. However, we're building a model for Hyperbolic Geometry, not Euclidean, and the sets are convex; as illustrated, the **h**-line $\overleftrightarrow{AB}$ does not intersect l between A and B.

As illustrated in the last two figures, we will have two main cases to consider in showing that $\mathbb{H}$ satisfies axiom 9 of Neutral Geometry. Taking l to be a line in $\mathbb{H}$, we have

Case 1: l appears as an **e**-ray, $l = \{a + iy\} \subset \mathbb{H}$ for some $a \in \mathbb{R}$, and

Case 2: l appears as an **e**-semicircle, $l = \{x + iy \in \mathbb{H} : (x - c)^2 + y^2 = r^2\}$, where $c + i0$ is the center of the **e**-semicircle representing l, and r its radius.

For the first case, l separates $\mathbb{H}$ into two sets defined as follows

$$\begin{aligned} H_1 &:= \{x + iy \in \mathbb{H} : x < a\} \\ H_2 &:= \{x + iy \in \mathbb{H} : x > a\}. \end{aligned}$$

These are the two sets which the axiom requires be convex such that, given two points A and B, if A lies in one set and B lies in the second then $\overline{AB}$ intersects l at some point C such that $A - C - B$. Now, taking two points A and B of $\mathbb{H}$ (we're not yet concerned with how they're related to H_1 and H_2), either $\overleftrightarrow{AB}$ appears as an **e**-ray or as an **e**-semicircle. If $\overleftrightarrow{AB}$ appears as an **e**-ray, it's clear that either $l = \overleftrightarrow{AB}$ or $l \cap \overleftrightarrow{AB} = \emptyset$, so if A and B are both in H_1 (or H_2) and $\overleftrightarrow{AB}$ appears as an **e**-ray, then $\overline{AB} \cap l = \emptyset$ as desired.

If, on the other hand, $\overleftrightarrow{AB}$ appears as an **e**-semicircle, then we can write $\overleftrightarrow{AB} = \{x + iy \in \mathbb{H} : (x - d)^2 + y^2 = s^2\}$ for some $d \in \mathbb{R}$ and $s > 0$. If we assume that $A, B \in H_1$, then, with $A = x_1 + iy_1$ and $B = x_2 + iy_2$, we can assume wlog that $x_1 < x_2$ so that $x_1 < x_2 < a$. It follows that if C is a point such that $A - C - B$, where $C = x_C + iy_C$, then $x_1 < x_C < x_2 < a$. Therefore, no point C with $A - C - B$ lies on l, so $\overline{AB} \cap l = \emptyset$.

This gives us, for our first case, that H_1, and by symmetry H_2, is convex. We still need that if $A \in H_1$ and $B \in H_2$, then $\overline{AB}$ intersects l at a point C with $A - C - B$. So, assuming that $A = x_1 + iy_1 \in H_1$ and $B = x_2 + iy_2 \in H_2$, we clearly have $\overleftrightarrow{AB}$ appearing as an **e**-semicircle. Since we can describe $\overline{AB}$ as

$$\overline{AB} = \{x + iy \in \mathbb{H} : y = \sqrt{s^2 - (x-d)^2}, x_1 < x < x_2\},$$

and because $x_1 < a < x_2$, it follows that there is some point $C = a + iy_C$ such that $A - C - B$ and $l \cap \overline{AB} = \{C\}$.

This takes care of our first case, so now consider the second; $l = \{x + iy \in \mathbb{H} : (x-c)^2 + y^2 = r^2\}$, where $c + i0$ is the center of the **e**-semicircle representing l, and r is its radius. This line l will separate the plane into sets

$$\begin{aligned} H_1 &:= \{x + iy \in \mathbb{H} : (x-c)^2 + y^2 < r^2\} \\ H_2 &:= \{x + iy \in \mathbb{H} : (x-c)^2 + y^2 > r^2\} \end{aligned}$$

Again, we'll take two points of $\mathbb{H}$, $A = x_1 + iy_1$ and $B = x_2 + iy_2$, which leads to two (sub-)cases.

(i) Either $\overleftrightarrow{AB}$ appears as an **e**-ray, $\overleftrightarrow{AB} = \{a + iy : y > 0\}$, or

(ii) $\overleftrightarrow{AB}$ appears as an **e**-semicircle, $\overleftrightarrow{AB} = \{x + iy \in \mathbb{H} : ((x-d)^2 + y^2 = s^2\}$ where $d + 0i$ is the center of the semicircle representing $\overleftrightarrow{AB}$ and s is its radius.

For the first case we'll assume that $y_2 > y_1$ and for the second we'll assume that $x_2 > x_1$, but in each case we'll parameterize $\overleftrightarrow{AB}$ and build a strictly increasing (or strictly decreasing or constant) function which is zero only at points of l.

For our first case, 2(i), we can parameterize $\overleftrightarrow{AB}$ with the function $f_1 : \mathbb{R} \to \overleftrightarrow{AB}$ defined by $f_1(t) = a + ie^t$. Since we're interested in the possible intersection of l and $\overleftrightarrow{AB}$, we'll use this parameterization to define another function,

$$g_1(t) := (a-c)^2 + e^{2t} - r^2.$$

Notice that if $g_1(t) = 0$, then l and $\overleftrightarrow{AB}$ intersect at $f_1(t_0)$. Also, $g_1'(t) = 2e^{2t}$, so g_1 is a strictly increasing function. Letting $f_1(t_1) = A$ and $f_1(t_2) = B$, we have that $t_1 < t_2$ since $y_1 < y_2$.

If we then assume $A, B \in H_1$, then it follows that $g_1(t_1) < g_1(t_2) < 0$. Since g_1 is strictly increasing, for any t such that $t_1 < t < t_2$ we will have $g_1(t) < 0$; therefore, there is no $t \in (t_1, t_2)$ such that $g_1(t) = 0$, and $\overline{AB} \cap l = \emptyset$. This means that H_1 is convex, and by a symmetric argument H_2 is also convex.

If we instead have that $A \in H_1$ and $B \in H_2$, then $g_1(t_1) < 0$ and $g_1(t_2) > 0$. The continuity of g_1 then implies that there is some t_0 such that $g_1(t_0) = 0$ and $t_1 < t_0 < t_2$. Therefore, if A and B are on different sides of l, then $\overline{AB}$ intersects l at a point C, $A - C - B$, where $f(t_0) = C$.

For case 2(ii) we'll need to recall from calculus the hyperbolic functions $\tanh(x)$ and $\operatorname{sech}(x)$. Taking

$$\overleftrightarrow{AB} = \{x + iy \in \mathbb{H} : ((x-d)^2 + y^2 = s^2\},$$

we will parameterize $\overleftrightarrow{AB}$ with f_2 and define a strictly increasing (or strictly decreasing or constant) function g_2. Parameterizing $\overleftrightarrow{AB}$ with $f_2(t) = d + s\tanh(t) + is\ \operatorname{sech}(t)$, and then defining g_2 by

$$g_2(t) = (d - c + s\tanh(t))^2 + (s\ \operatorname{sech}(t))^2 - r^2,$$

we have that if $g_2(t') = 0$ then l and $\overleftrightarrow{AB}$ intersect at $f_2(t')$, and that

$$\begin{aligned} g_2'(t) &= 2(d - c + s\tanh(t))(s\ \operatorname{sech}^2(t)) + 2s\ \operatorname{sech}(t)(-s\tanh(t)\operatorname{sech}(t)) \\ &= 2(d-c) \end{aligned}$$

Therefore, g_2 is either strictly increasing (if $d > c$), strictly decreasing (if $d < c$), or constant (if $d = c$); notice that if g_2 is constant, then either $\overleftrightarrow{AB}$ and l never intersect, or $\overleftrightarrow{AB}$ and l are the same line. Then, again letting $f_2(t_1) = A$ and $f_2(t_2) = B$, assume wlog that $A, B \in H_1$ and that g_2 is increasing so that $g_2(t_1) < g_2(t_2) < 0$. As before, the monotonicity of g_2 implies that no $t \in (t_1, t_2)$ gives $g_2(t) = 0$, and therefore no point C exists such that $A - C - B$ and $\overleftrightarrow{AB}$ intersects l and C. Therefore, H_1 is convex; a symmetric argument also shows that H_2 is convex.

Furthermore, if $A \in H_1$ and $B \in H_2$, then $g_2(t_1) < 0 < g_2(t_2)$. The continuity of g_2 then implies that there is some $t_0 \in (t_1, t_2)$ such that $g_2(t_0) = 0$, so that $A - f_2(t_0) - B$ and $\overleftrightarrow{AB} \cap l = \{f_2(t_0)\}$.

Having covered each case (and sub-case), we have that $\mathbb{H}$ satisfies axiom 9 of Neutral Geometry.

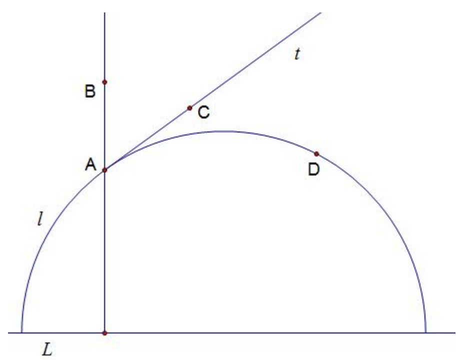
Now, skipping over axiom 10 (since it deals with three dimensions), we need to discuss angles in this model before we can visit the eleventh axiom. Taking $P, Q, R \in \mathbb{H}$, we will separate this into cases since we have lines appearing differently. First, suppose that $\mathbf{h}\overleftrightarrow{PQ}$ and $\mathbf{h}\overleftrightarrow{QR}$ appear as an **e**-ray and an **e**-semicircle respectively. If t is the **e**-tangent to $\overleftrightarrow{QR}$ as an **e**-semicircle at Q, then let R' be a point on t, on the same side of $\overleftrightarrow{PQ}$ as R. Then define $m_h\angle PQR = m_e\angle PQR'$. If, on the other hand, both $\mathbf{h}\overleftrightarrow{PQ}$ and $\mathbf{h}\overleftrightarrow{QR}$ appear as **e**-semicircles, then take t and s to be **e**-tangents to $\overleftrightarrow{PQ}$ and $\overleftrightarrow{QR}$ respectively, each through Q. Take a point R' on s, and P' on t, appearing on the same side of $\overleftrightarrow{PQ}$ and $\overleftrightarrow{QR}$, respectively, as their counterparts. Then define the hyperbolic measure of angle $\angle PQR$ to be the euclidean measure of angle $\angle P'QR'$, $m_h\angle PQR = m_e\angle P'QR'$.

Axiom 11: To every angle there corresponds a real number between $0°$ and $180°$.

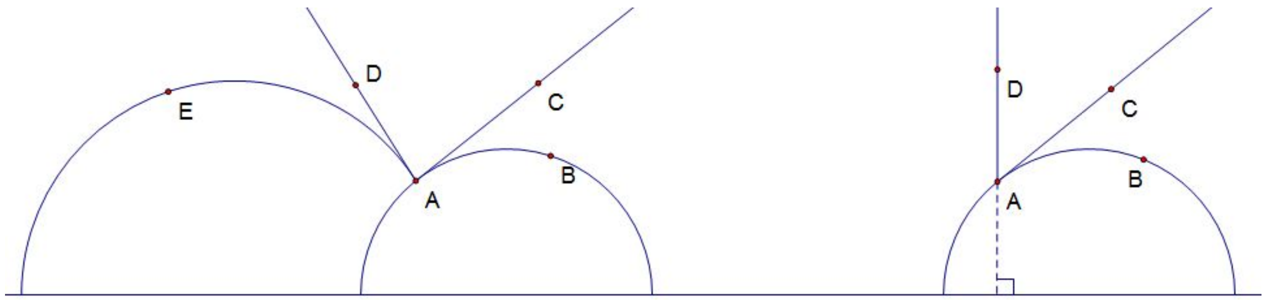
By the way we have defined the measure of an angle in our model, we have that every angle can be measured in a Euclidean fashion. So just as $\mathbb{C}$ satisfies this axiom as a model of Euclidean geometry, $\mathbb{H}$ satisfies this axiom as a model of Hyperbolic Geometry.

Axiom 12: Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every R between 0° and 180°, there is exactly one ray $\overrightarrow{AP}$ with P in H such that $m, PAB = r$.

First let us take an **h**-line $\overleftrightarrow{AB}$, and assume that this appears as an **e**-ray. Then for any r, with $0 < r < 180$, there is unique **e**-line t passing through A, with point C on a desired side of $\overleftrightarrow{AB}$ on t such that $m_e\angle BAC = r$. We want this line t to be tangent to an **h**-line l, at A, and a result regarding circles gives us the existence and uniqueness of such a line. Then choosing any point D of l, on the same side of $\overleftrightarrow{AB}$ as C, we have the unique ray $h\overrightarrow{AD}$ with $m_h\angle BAD = m_e\angle BAC = r$.



This takes care of the case in which the given line appear as an **e**-ray, so now assume that we start with $\overleftrightarrow{AB}$ appearing as an **e**-semicircle, and take $\overleftrightarrow{AC}$ to be the **e**-line tangent to $\overleftrightarrow{AB}$ at A. Then for any r with $0 < r < 180$, there is a unique **e**-ray $\overrightarrow{AD}$ with $m_e\angle CAD = r$. Then we have two cases: if $\overrightarrow{AD}$ is vertical, then this is also an **h**-ray giving us the desired angle measure, and if $\overrightarrow{AD}$ is not a vertical ray then, as before, we are guaranteed the existence and uniqueness of an **e**-semicircle passing through A, **h**$\overleftrightarrow{AE}$, with $\overrightarrow{AD}$ as a tangent. Then, assuming that E is on the same side of $\overleftrightarrow{AB}$ as C, we have that $m_h\angle BAE = m_e\angle CAD = r$.

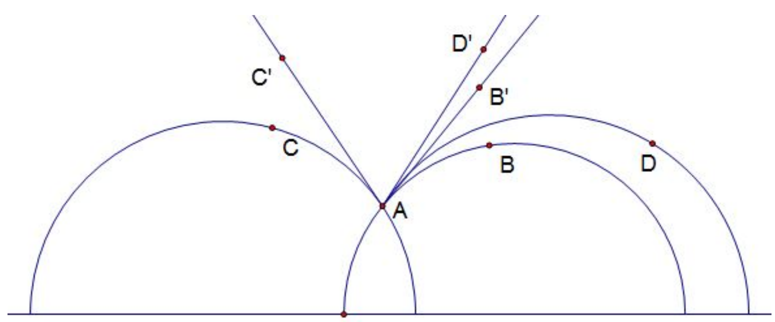


Axiom 13: If D is a point in the interior of $\angle BAC$, then

$$m\angle BAC = m\angle BAD + m\angle DAC.$$

To begin, consider an angle $\angle BAC$, with the corresponding **h**-lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, and take D to be a point on the interior of $\angle BAC$. Then, with our measure of angles, let $m_h\angle BAC = m_e\angle B'AC'$ where B' and C' are points on tangent lines (where necessary) corresponding to B and C respectively. Then, let D' correspond to D (on the **e**-tangent to $\mathbf{h}\overleftrightarrow{AD}$ if necessary), so that $m_h\angle CAD = m_e\angle C'AD'$ and $m_h\angle DAB = m_e\angle D'AB'$. Since D' is interior to $\angle CAB$, we have that

$$m_h\angle CAB = m_e\angle C'AB' = m_e\angle C'AD' + m_e\angle D'AB' = m_h\angle CAD + m_h\angle DAB.$$



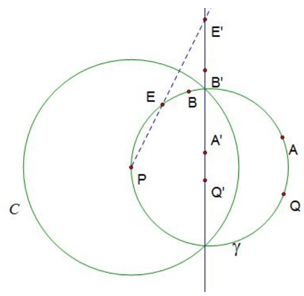
Axiom 14: If two angles form a linear pair, then they are supplementary.

Take an **h**-line $\overleftrightarrow{AB}$, a point D on $\overleftrightarrow{AB}$ such that $D - A - B$, and an **h**-ray $\overrightarrow{AC}$. Then take **e**-ray $\overrightarrow{AC'}$ (tangent to $\mathbf{h}\overrightarrow{AC}$ if necessary), and let $\mathbf{e}\overleftrightarrow{AB'}$ be the **e**-line tangent to $\overleftrightarrow{AB}$ (if necessary). Finally, let D' be a point on $\mathbf{e}\overleftrightarrow{AB'}$ such that $D' - A - B'$. All of this gives us that $m_h\angle DAC = m_e\angle D'AC'$ and $m_h\angle CAB = m_e\angle C'AB'$. Since $\mathbf{e}\angle D'AC'$ and $\mathbf{e}\angle C'AB'$ form a linear pair, they are supplementary, and since these two Euclidean angles are supplementary, so are their Hyperbolic counterparts, $\mathbf{h}\angle DAC$ and $\mathbf{h}\angle DAB$.

This completes axiom 14, and before we move on to the last axiom, that of congruence, consider that Theorem 2.21 (ASA Congruence) is equivalent to Axiom 15. We already have that the side-angle-side congruence condition implies the angle-side-angle congruence condition, so to prove their equivalence assume the angle-side-angle congruence condition.

Given triangles ΔABC and ΔXYZ in which $\overline{AB} \cong \overline{XY}$, $\angle ABC \cong \angle XYZ$, and $\overline{BC} \cong \overline{YZ}$, if $\angle BCA \cong \angle YZX$ then we have congruent triangles by our hypothesis, so assume that these two angles are not congruent. In particular, let $m\angle BCA > m\angle YZX$. Then by the Cross-Bar Theorem there is a point D with $A - D - B$, and $\angle BCD \cong \angle YZX$. Then we have $\Delta BCD \cong \Delta YZX$, and in particular $\overline{DB} \cong \overline{XY} \cong \overline{AB}$, so $A = D$. Thus, $\Delta ABC \cong \Delta XYZ$, and we have that the side-angle-side congruence condition for triangles holds assuming the angle-side-angle congruence condition.

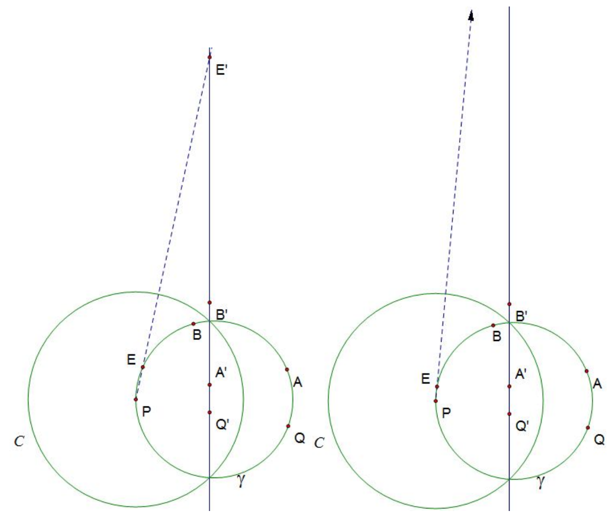
Another idea we'll need is that inversion through a circle preserves hyperbolic distance. We already know that inversion preserves the ratio $\frac{|AP||BQ|}{|AQ||BP|}$ by Theorem 4.21 when none of the points A, B, P, and Q are the center of the circle of inversion. So if γ is an **h**-line appearing as a semicircle, and if $\overleftrightarrow{AB}$ is an **h**-line appearing as a semicircle such that neither ideal point of $\overleftrightarrow{AB}$ is the center of the **e**-semicircle γ, then $d_{\mathbb{H}}(A,B) = d_{\mathbb{H}}(A',B')$, where $A' = I_\gamma(A)$ and $B' = I_\gamma(B)$. However, what if $\overleftrightarrow{AB}$, still appearing as an **e**-semicircle, has the center of the **e**-semicircle γ as an ideal point? Or, what if $\overleftrightarrow{AB}$ appears as an **e**-ray, and its ideal point is the center of the **e**-semicircle γ? For the first of these questions we'll move out of $\mathbb{H}$ and look at a slightly more general situation.



Now. take points A, B, P, and Q to lie on a circle γ, and let P be the center of circle C. Since the inverse of P through C is not defined, we'll take another point E on the circle γ to vary across γ so that $E \to P$. Recall that the image of γ under inversion through C will be a line, and take $I_C(A) = A'$, $I_C(B) = B'$, $I_C(Q) = Q'$, and $I_C(E) = E'$, and assume that $Q' - A' - B' - E'$. Now, as $E \to P$, we have

$$\begin{aligned}
\lim_{E\to P} \frac{|AE||BQ|}{|AQ||BE|} &= \lim_{E\to P} \frac{|A'E'||B'Q'|}{|A'Q'||B'E'|} \\
&= \frac{|B'Q'|}{|A'Q'|} \lim_{E\to P} \frac{|A'E'|}{|B'E'|} \\
&= \frac{|B'Q'|}{|A'Q'|} \lim_{E\to P} \frac{|A'B'| + |B'E'|}{|B'E'|} \qquad (5.2) \\
&= \frac{|B'Q'|}{|A'Q'|} \lim_{|B'E'|\to\infty} \frac{|A'B'| + |B'E'|}{|B'E'|} \qquad (5.3) \\
&= \frac{|B'Q'|}{|A'Q'|} \lim_{|B'E'|\to\infty} \frac{1}{1} \\
&= \frac{|B'Q'|}{|A'Q'|}
\end{aligned}$$

Our transition from line 5.2 to line 5.3 is made by observing that as E approaches P, the distance from B', which we're leaving fixed, to E' tends towards infinity.



This takes care of the first question; inversion preserves distance when mapping from an **e**-semicircle to an **e**-ray. Notice that, because $I_C^{-1} = I_C$ for any circle C, this also means that inversion preserves distance when mapping a segment from an **e**-ray to an **e**-semicircle. For the second question, we'll return to $\mathbb{H}$ since the distance function $d_{\mathbb{H}}$ will be needed explicitly.

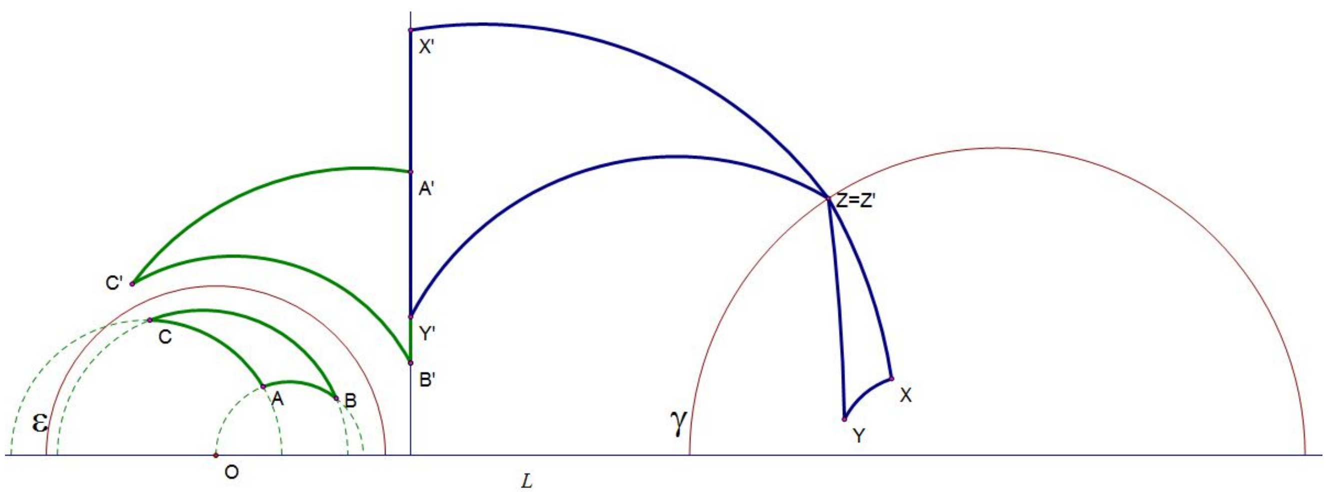
We'll let γ be a line appearing as an **e**-semicircle with center O, and let l be a line with ideal point O and appearing as an **e**-ray. Then take points A and B of l such that $O - A - B$ (in the Euclidean sense). Then our goal will be to show that

$$d_{\mathbb{H}}(A, B) = \left| \ln \frac{|OB|}{|OA|} \right| = \left| \ln \frac{|OB'|}{|OA'|} \right| = \ln \frac{|OA'|}{|OB'|} = d_{\mathbb{H}}(A', B').$$

Since $|OA||OA'| = r^2 = |OB||OB'|$ where r is the radius of **e**-semicircle γ, it follows that $\frac{|OB|}{|OA|} = \frac{|OA'|}{|OB'|}$. This gives us the desired result.

Axiom 15: If two sides and the included angle of a triangle are congruent to the corresponding parts of a second triangle, then the correspondence is a congruence.

By the previous discussion, we need only show that the angle-side-angle congruence condition for triangles holds in our model. So, let ΔABC and ΔXYZ be two h-triangles such that $\angle CAB \cong \angle ZXY$, $\overline{AB} \cong \overline{XY}$, and $\angle CBA \cong \angle ZYX$. Without loss of generality, we may assume that each of the h-lines creating these triangles have two ideal points. In particular, let O be one ideal point of $\overleftrightarrow{AB}$. Then, letting ϵ be a circle with center O, inversion through ϵ sends ΔABC to a triangle $\Delta A'B'C'$ where $\overleftrightarrow{A'B'}$ is a segment of an h-line l with a single ideal point P. Then, by Theorem 4.33, there is a circle γ through which inversion sends ΔXYZ to $\Delta X'Y'Z'$ where $\overline{X'Y'}$ is a segment of l. Since inversion preserves hyperbolic length and the measure of angles between circles, and between a circle and a line, we have that $\Delta ABC \cong \Delta A'B'C'$, and $\Delta XYZ \cong \Delta X'Y'Z'$, and the congruences between corresponding angles and included side still hold for these new triangles.



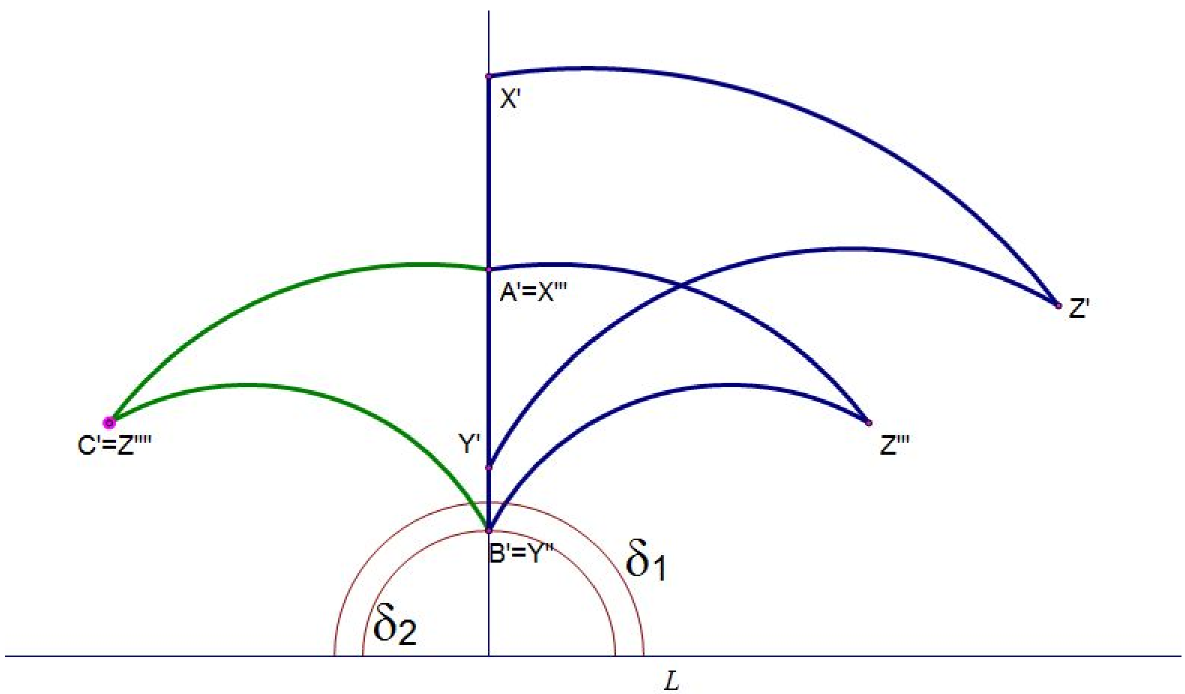
Now, there is a circle δ_1 with center P, through which inversion sends Y' to B' (and X' to X'', Z' to Z''); the construction of such δ_1 is similar to the constructions discuss in remarks 7 and 8. Since inversion preserves hyperbolic distance we know that $\overline{A'B'} \cong \overline{X'Y'} \cong \overline{X''B'}$, so a circle δ_2 with center P and radius $|PB'|$ will map

X'' to A' (and Z'' to Z''').

This gives us two triangles $\Delta A'B'C'$ and $\Delta A'B'Z'''$, in which

$$\angle B'A'C' \cong \angle B'A'Z''' \text{ and } \angle A'B'C' \cong \angle A'B'Z'''.$$

Supposing that Z' and C' are on different sides of l, we may reflect $\Delta A'B'Z'''$ across l to attain $\Delta A'B'Z''''$ with the same congruences (if Z' and C' are on the same side of l, then Z'''' need only be replaced with Z''' in what follows).



This gives us two triangles $\Delta A'B'C'$ and $\Delta A'B'Z''''$ which we must show are congruent. By axiom 12, and because $\angle A'B'C' \cong \angle A'B'Z''''$, Z'''', C', and A' must be collinear. Similarly, Z'''', C', and B' must be collinear. Then, in accordance with axiom 1, it must be that $Z'''' = C'$. The following congruences then hold:

$$\overline{AC} \cong \overline{A'C'} \cong \overline{A'Z''''} \cong \overline{X'Z'} \cong \overline{XZ}$$

$$\overline{BC} \cong \overline{B'C'} \cong \overline{B'Z''''} \cong \overline{Y'Z'} \cong \overline{YZ}$$

$$\angle ACB \cong \angle A'C'B' \cong \angle A'Z''''B' \cong \angle X'Z'Y' \cong \angle XZY.$$

Thus, the angle-side-angle congruence condition for triangles holds in our model, and it follows that the side-angle-side congruence condition also holds.

Finally, we can address the last axiom of Hyperbolic Geometry.

The Hyperbolic Axiom: There exists a line l, and a point P not on l, such that two distinct lines exist which are parallel to l and pass through P.

Consider a line l appearing as an e-semicircle, with ideal points A and B. Now let t and s be two more lines appearing as e-semicircles, where A and C are ideal points of t, and B and D are ideal points of s, with $A-D-C-B$. By this ordering, t and s must intersect at a point P, and because $|AC| < |AB|$ and $|BD| < |AB|$ both t and s are parallel to l. This construction satisfies our last axiom, and $\mathbb{H}$ is therefore a valid model for Hyperbolic Geometry.

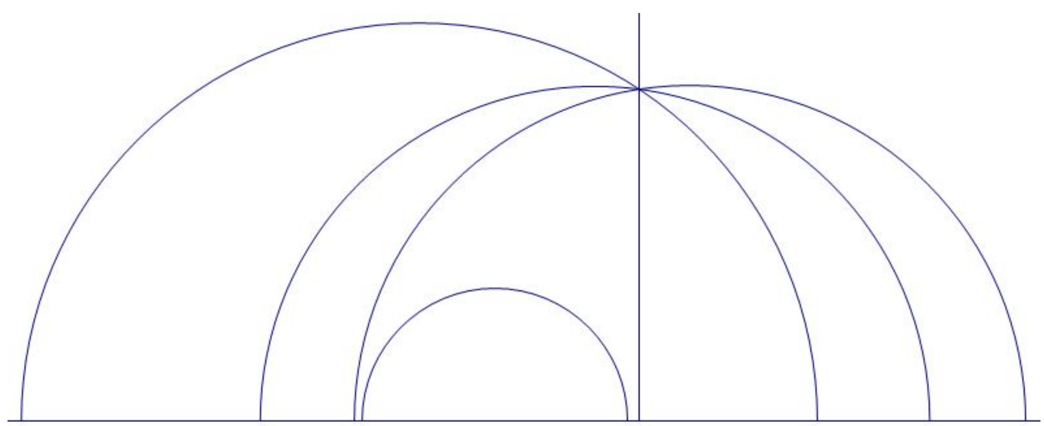


Figure 5.3: Illustration of the Hyperbolic Axiom in $\mathbb{H}$.

Chapter 6

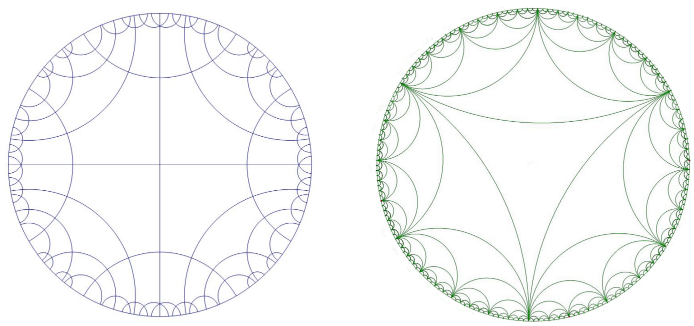
Models of Hyperbolic Geometry

6.1 History

In the last chapter, we explored one model of geometry: the half-plane. This model is often credited to Poincaré, possibly due to the wide array of mathematical fields he helped to advance, and his extensive use of Hyperbolic Geometry (and its established models) in a variety of fields including complex analysis, number theory, and differential equations. However, the half-plane model was actually developed by Beltrami (1868), aided by results from Liouville. In particular, Liouville had discovered a constant curvature metric, obtained through transformations of the line element of the pseudosphere [4]. In fact, Beltrami is largely responsible for two of the models we see in this chapter, and it is for this reason that Stillwell refers to the half-plane, conformal (or Poincaré) disk, and Klein disk models as the Liouville-Beltrami, Riemann-Beltrami, and Cayley-Beltrami models respectively.

These three models were developed by Beltrami, at least in part, to show the equiconsistency of the Hyperbolic and Euclidean Geometries. Beginning with the hemisphere model, which we'll see very soon, Beltrami developed these other three models, not just for two dimensions, but for general n-dimensional Hyperbolic Geometry.

6.2 Poincaré Disk $\mathfrak{P}$

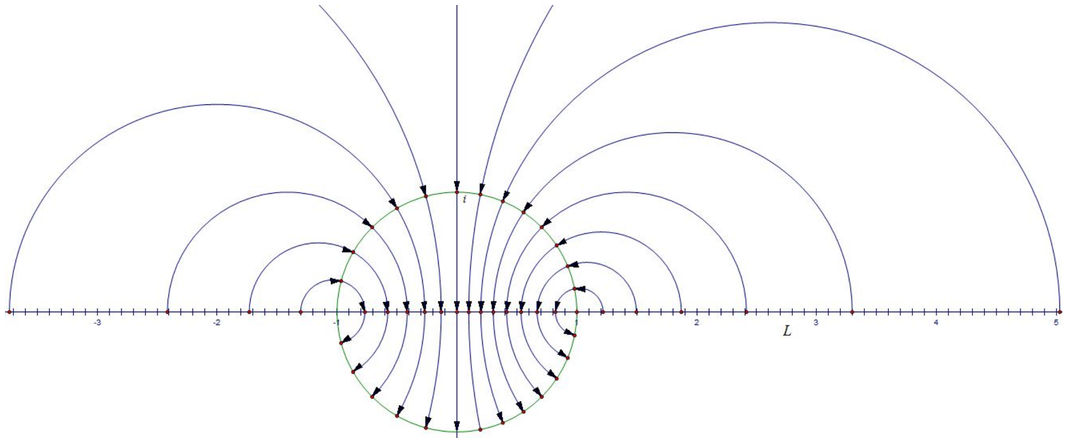


6.2.1 Building the Model

The next model to visit is the Poincaré disk model, sometimes called the conformal disk model. To move into this model of Hyperbolic Geometry, we'll need to use some of the Euclidean results previously established; we'll be using a möbius transformation to bring the half-plane into the unit disk. In particular, we want a transformation $m(z) = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$, $m(0) = -i$, $m(\infty) = i$, and $m(i) = 0$. Given what we know of möbius transformations, we can then assume that $\frac{b}{d} = -i$, $\frac{a}{c} = -i$, and $ai + b = 0$. This gives us the transformation

$$m(z) = \frac{-z+i}{zi-1}.$$

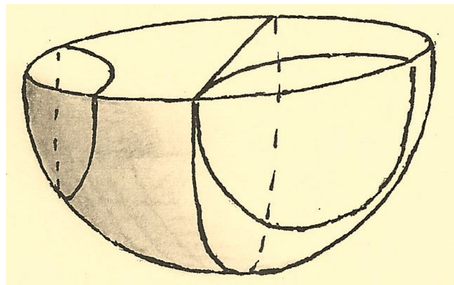
Note that we may use this function's inverse to move back from Poincaré's disk into the half-plane: $m^{-1}(z) = \frac{z+i}{zi+1}$.



Moving to this model is this way gives us a good deal of information. For instance, because the half-plane model is conformal, and möbius transformations are conformal mappings of the extended complex plane to itself, Poincaré's disk is also conformal. Also, because möbius transformations send lines and circles to lines and circles, and because **h**-lines appear in the half-plane as Euclidean semi-circles or rays, **h**-lines will appear in this model to be circular arcs or line segments; in particular, lines will appear as the arcs of circles orthogonal to the unit disk or as diameters of the disk. Notice that, because our möbius transformation sent i to 0, an **h**-line will appear as a diameter of the disk iff its half-plane representation passes through i.

Another advantage this approach provides comes from the fact that möbius transformations preserve the ratio we used in the half-plane model to define distance. Recall that, by Theorem 4.39, if f is a möbius transformation and A, B, P, and Q are points of the extended complex plane, then $\frac{|AP||BQ|}{|AQ||BP|} = \frac{|f(A)f(P)||f(B)f(Q)|}{|f(A)f(Q)||f(B)f(P)|}$. This lets us define distance in the Poincaré disk between points A and B to be $d_P(A,B) = \ln\left|\frac{|AP||BQ|}{|BP||AQ|}\right|$, where, as before, P and Q are ideal points of $\overleftrightarrow{AB}$; though the ideal points now make up the boundary of the disk, since m sends L to ∂S^1. By its relation to the half-plane model, and what we know about möbius transformations, the Poincaré disk clearly satisfies the axioms of Hyperbolic Geometry.

6.3 Hemisphere $\mathfrak{H}$



6.3.1 Building the Model

This is the model Beltrami started with in his development of the other three (the last of which appears in the next section). It should make sense, then, that this model will be the intermediary step between our two disk models.

In order to move from the Poincaré disk to the hemisphere, we'll build a function f which projects from the north pole of $S^2 = \{(z,t) \in \mathbb{C} \times \mathbb{R} : z\bar{z} + t^2 = 1\}$ onto the southern hemisphere of S^2; note that the choice of S^2 (as well our choice of using the southern hemisphere) is arbitrary, and is simply a convenient choice since we've used the unit disk for the setting of the Poincaré disk. Here we'll use the fact that a line passing through the point $(x, y, 0)$ (or $(z, 0)$) and $(0, 0, 1)$ is given by

$$w(x, y, 0) + (1 - w)(0, 0, 1) = (wx, wy, 1 - w)$$

and letting w vary across $\mathbb{R}$.

Now, we're interested in projecting the Poincaré disk (the unit disk) onto the southern hemisphere of S^2, so for any $(x, y) \in S^1$ we want to find w such that $w(x, y, 0) + (1 - w)(0, 0, 1)$ intersects the southern hemisphere; then we're looking at

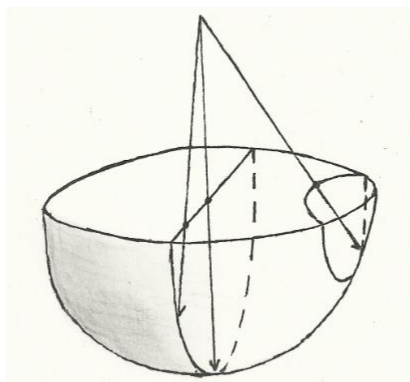
$$w^2(x^2 + y^2) + (1 - w)^2 = 1$$

with $1 - w < 0$. This leads us to the equation $w^2(x^2 + y^2 + 1) - 2w = 0$, which yields solutions $w = 0$ and $w = \frac{2}{x^2+y^2+1} > \frac{2}{2} = 1$. Since the second of these solutions matches our requirement that $1 - w < 0$, this is what we'll use so that the point which $(x, y, 0)$ will be projected to is

$$\frac{2}{x^2 + y^2 + 1}(x, y, 0) + \left(1 - \frac{2}{x^2 + y^2 + 1}\right)(0, 0, 1) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\right),$$

or $\left(\frac{2z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$. So we can define $f : \mathbb{C} \to \mathbb{C} \times \mathbb{R}$ by

$$f(z) = \left(\frac{2z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right).$$



We should be able to move back and forth between models, and the following function would move us back from the hemisphere model to the Poincaré disk:

$$f^{-1}(z, t) = \frac{z}{1 + \sqrt{1 - |z|^2}}.$$

It's important to see how we can relate the image of a line l from the Poincaré disk to it's image under f. First off, if l is a line appearing as a diameter of the disk, then it's image under f will be one half of a great circle of S^2 passing through the south pole; note that we can view this as the intersection of the southern hemisphere with a plane in $\mathbb{C} \times \mathbb{R}$ orthogonal to $\mathbb{C}$ and passing through the origin. As for the rest of our lines in the Poincaré disk, those appearing as arcs of circles orthogonal to the unit disk, these will be sent to the intersection of a sphere with the southern hemisphere of S^2. More specifically, given a line l of $\mathfrak{P}$ described by a Euclidean circle α with center $\omega = (\omega_1, \omega_2)$, radius r, and orthogonal to the disk, the image of l under f will be the intersection of the southern hemisphere with the sphere α', which has center $(\omega, 0)$ and radius r. This fact may not be clear, so to confirm it take line l and circle α to be as described. We have that

$$|f(z)|^2 = \left(\frac{2z}{z\bar{z}+1}\right)\left(\frac{2\bar{z}}{z\bar{z}+1}\right) + \left(\frac{z\bar{z}-1}{z\bar{z}+1}\right)^2 = 1$$

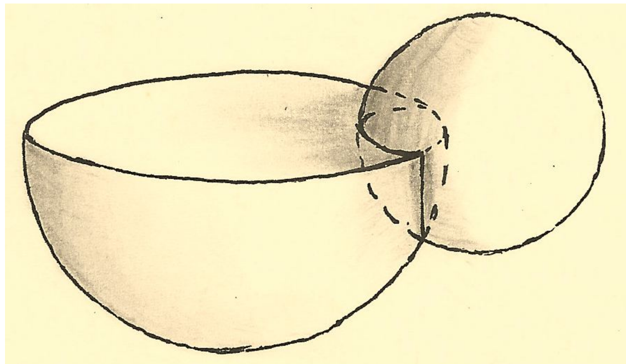
for any $z \in l$ by virtue of the fact that z is being sent to the surface of S^2 by f. Then we need that

$$\left(\frac{2z}{z\bar{z}+1} - \omega\right)\left(\frac{2\bar{z}}{z\bar{z}+1} - \bar{\omega}\right) + \left(\frac{z\bar{z}-1}{z\bar{z}+1}\right)^2 = r^2$$

where r is the radius of α. To confirm this, we'll use that the inverse of z through S^1 is $z' = \frac{1}{\bar{z}}$.

$$\begin{aligned}
\left(\frac{2z}{z\bar{z}+1} - \omega\right)\left(\frac{2\bar{z}}{z\bar{z}+1} - \bar{\omega}\right) + \left(\frac{z\bar{z}-1}{z\bar{z}+1}\right)^2 &= 1 - \frac{2z\bar{\omega} + 2\bar{z}\omega}{z\bar{z}+1} + \omega\bar{\omega} \\
&= \frac{z\bar{z} + 1 + \omega\bar{\omega}z\bar{z} - 2z\bar{\omega} - 2\omega\bar{z}}{z\bar{z}+1} \\
&= \frac{r^2 + z\bar{z}\left(\omega\bar{\omega} - \frac{\omega}{z} - \frac{\bar{\omega}}{\bar{z}} + \frac{1}{z\bar{z}}\right)}{z\bar{z}+1} \\
&= \frac{r^2 + z\bar{z}(\omega\bar{\omega} - \omega\bar{z}' - \bar{\omega}z' + z'\bar{z}')}{z\bar{z}+1} \\
&= \frac{r^2(z\bar{z}+1)}{z\bar{z}+1} \\
&= r^2
\end{aligned}$$

This tells us that the image of l under f is the desired intersection.



This relationship is helpful for a couple reasons, the first being that it will serve as a stepping stone to our next model. The second, and perhaps more interesting reason is that is gives us a view into three dimensional hyperbolic space. Recall axiom 8: if two distinct planes intersect, then their intersection is a line. What we have with this hemisphere is a model of two-dimensional Hyperbolic Geometry, so our hemisphere is a Hyperbolic plane on which lines are given by the intersections of S^2 with both Euclidean half-planes perpendicular to $\mathbb{C}$, and Euclidean (southern) hemispheres with centers on $\mathbb{C}$. These Euclidean half-planes and hemispheres are actually other Hyperbolic planes. From this, we can get a vague idea of how to visualize a model of three dimensional Hyperbolic Geometry in $\mathbb{C} \times \mathbb{R}^-$, or $\mathbb{C} \times \mathbb{R}^+$ if we changed our discussion to northern hemispheres, and it helps us relate the hemisphere back to the half-plane model we started with.

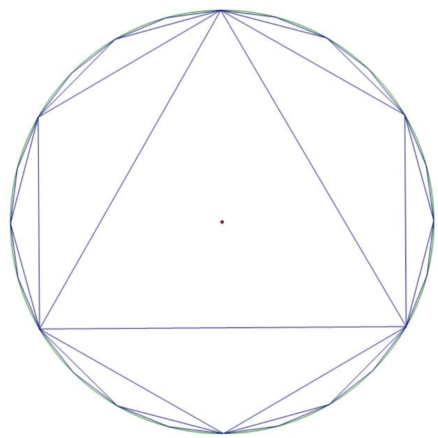
Now, we can actually look at the intersections of hemispheres with the southern hemisphere to S^2 as the intersection of half-planes with S^2 instead, and this will lead to our next model for Hyperbolic Geometry.

6.3.2 An Alternate Approach to $\mathfrak{H}$

In the previous section, our approach to moving into $\mathfrak{H}$ was the inverse of stereographic projection from the north pole of S^2. An alternate approach we can take involves three dimensional inversion: inversion through spheres. In particular, consider the sphere S whose center is the north pole of S^2 and whose radius is $\sqrt{2}$. The intersection of S with $\mathbb{C}$ is S^1, the unit circle. Let inversion through S be denoted I_S, and recall what we know about circle inversion. In particular, when inverting through a circle γ, if a second circle δ passes through the center of γ then inversion through γ maps δ to a line. We have a similar situation for inversion through spheres. Since S^2 passes through the center of S, I_S will map S^2 to a plane; specifically, $I_S(S^2) = \mathbb{C}$ and $I_S(\mathbb{C}) = S^2$.

Now, any circle C orthogonal to S^1 is the equator of a sphere S_C whose center is on $\mathbb{C}$. Furthermore, such a sphere S_C is orthogonal to $\mathbb{C}$, S^2, and S. Since this is the case, $I_S(S_C) = S_C$, so $I_S(S_C \cap \mathbb{C}) = S_C \cap S^2$; in particular, I_S will map the intersection of S_C with the unit disk to the intersection of S_C with the southern hemisphere of S^2. Notice that the intersection of S_C with the unit disk is a line in $\mathfrak{P}$ and that the intersection of S_C with the southern hemisphere of S^2 is a line in $\mathfrak{H}$. Essentially, we have that $I_S|_{\mathbb{C}}$, the restriction of I_S to $\mathbb{C}$, is the function f we defined earlier, and restricting each of these to $\mathfrak{P}$ results in $\mathfrak{H}$.

6.4 Klein Disk $\mathfrak{K}$



6.4.1 Building the Model

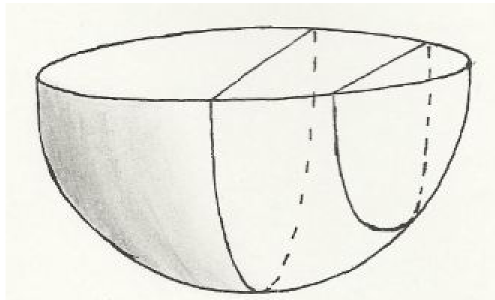
As mentioned in the previous section, lines in $\mathfrak{H}$ are the intersection of spheres, whose centers on $\mathbb{C}$ and which are orthogonal to S^2, with the southern hemisphere of S^2. These intersections will be semi-circles on S^2 which are orthogonal to the equator and lie on planes orthogonal to $\mathbb{C}$ (as noted at the end of the previous section). We can use these ideas to move into yet another model of Hyperbolic Geometry.

This new model will be constructed using the simple projection function $g : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ defined by $g(z,t) = z$. Now, consider a line l in the hemisphere model whose image under f^{-1} is l', a line in the Poincaré disk appearing as an arc of a Euclidean circle with center $\omega = (\omega_1, \omega_2)$ and radius r. Then l lies on both S^2 and the sphere with center $(\omega, 0)$ and radius r, given by

$$(z-\omega)(\bar{z}-\bar{\omega}) + t^2 = r^2.$$

It follows that, for any $(z,t) \in l$, we have

$$(z-\omega)(\bar{z}-\bar{\omega}) + t^2 = r^2, \quad \text{and} \quad \bar{z}z + t^2 = 1.$$



This leads us to

$$\begin{aligned}-\omega\bar{z} - \bar{\omega}z + \omega\bar{\omega} + 1 &= r^2 \qquad \text{or} \\ -2\omega_1 x - 2\omega_2 y + 1 &= r^2\end{aligned}$$

which is a linear equation of x and y. Clearly now we have that the image of l, an **h**-line appearing as an arc, will be a euclidean line segment under g. However, what if l' instead appears as a diameter? In this case, note that we can represent l' by a linear equation $ax'+by'=0$ (ie: $l' = \{z' = (x'+iy') \in \mathbb{C} : ax'+by'=0,\ and\ x'^2+y'^2<1\}$) for some $a, b \in \mathbb{R}$. Then for any $z' \in l'$,

$$f(z') = f((x',y')) = \left(\frac{2x'}{x'^2+y'^2+1}, \frac{2y'}{x'^2+y'^2+1}, \frac{x'^2+y'^2-1}{x'^2+y'^2+1}\right) = (x,y,t) = (z,t),$$

and

$$g(z,t) = z = \left(\frac{2x'}{x'^2+y'^2+1}, \frac{2y'}{x'^2+y'^2+1}\right).$$

It follows from $ax'+by'=0$ that

$$\frac{2ax'}{x'^2+y'^2+1} + \frac{2by'}{x'^2+y'^2+1} = \frac{2}{x'^2+y'^2+1}(ax'+by') = 0,$$

so a line in $\mathfrak{P}$ appearing as a diameter will be sent to itself by the composition $g \circ f$.

What all this gives us is a function $\phi : \mathbb{C} \to \mathbb{C}$ defined by $\phi(z) = (g \circ f)(z)$, for the function f defined in the previous section. This function ϕ will send a line l from the Poincaré disk to the chord of the unit disk whose endpoints are the ideal points of l (these ideal points are clearly fixed by f). Furthermore, the image of the Poincaré disk under ϕ is the model of Hyperbolic Geometry for which this section is named: the Klein model. This means that lines will be represented in the Klein model as chords, and the endpoints of such a chord will be ideal points of the line it represents.

6.4.2 History of the Model

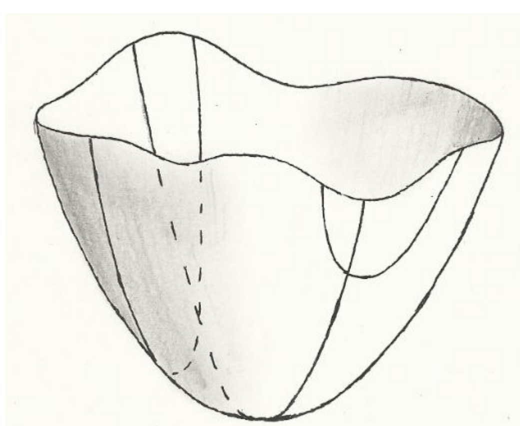
This model first appeared, not directly as a model of Hyperbolic Geometry, but in relation to projective geometry in Cayley's work (1859) (this should make sense considering how we moved form the hemisphere to the Klein disk). It was Beltrami who first connected our model to Hyperbolic Geometry in his 1868 paper "Fundamental theory of spaces of constant curvature." However, in 1928 Klein gives credit to Caley for the model, and claimed that he, Klein, had first realized its connection to Hyperbolic Geometry.

Before developing this model, Beltrami had been investigating the pseudosphere; as a surface of constant negative curvature the pseudosphere can me made to model *part* of the hyperbolic plane. However, the pseudosphere could not serve as a model of the entire hyperbolic plane as it has a boundary curve and is not simply connected. To get around these issues, Beltrami considered the universal cover of the pseudosphere, "a surface wrapped infinitely many times around the pseudosphere" [4], and removed the boundary curve of this covering. Beltrami had focused on the pseudosphere because he had wanted to find a surface, based in Euclidean Geometry, which could be used to model Hyperbolic Geometry and had "the ordinary notion of lengths of curves"[4]. However, the pseudosphere is only a special case of what Beltrami called a "pseudospherical surface," meaning an arbitrary surface of constant negative curvature given by the formula for its line element.

In 1850 Liouville had already found such a surface which *is* simply connected, the upper half plane with line element $\sqrt{dx^2 + dy^2}/y$, by transforming coordinates of the pseudosphere. We've seen the model of Hyperbolic Geometry this admits, but Beltrami was also interested in finding a model in which the lines of Hyperbolic Geometry would appear as euclidean lines. Investigating this, Beltrami had already known that central projection from a sphere (or hemisphere) onto a tangent plane mapped geodesics to euclidean lines, and made the discovery that only surfaces of constant curvature admit such a mapping. Applying this idea to the pseudosphere led Beltrami to what we call the Klein (or projective) model of Hyperbolic geometry. Stillwell notes that Beltrami's construction gave two advantages:

- first, that the hyperbolic plane is the open unit disk where lines appear as open chords of the disk, and
- second, that isometries of the plane are the projective transformations of the euclidean plane which map the unit disk to itself.

6.5 The Hyperboloid H^2



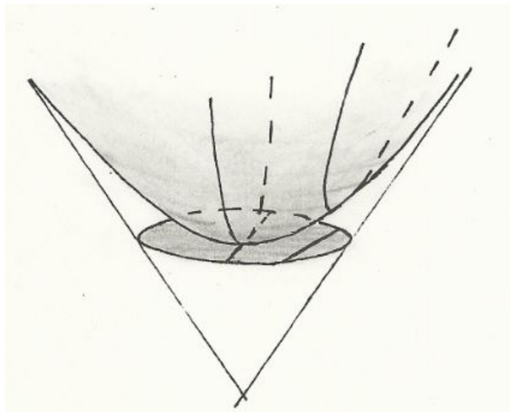
6.5.1 Building the Model

The final model we'll be moving into comes from physics. Here, we move the Klein disk into the plane $(z,t) = (z,1)$ so that the center of the disk is now $(\mathbf{0},1)$, and from the origin we want to project the Klein disk onto the upper-half of the two sheeted hyperboloid, given by $z\bar{z} - t^2 = -1$. Consider the equation of a line through the origin and any point interior to the disk, z_0: $z = tz_0$. This line will intersect the hyperboloid at two points, but we want the t-coordinate to be positive. In order to find an appropriate function, sending the Klein disk onto the desired surface, we can make the appropriate substitution into the equation describing the surface and solve for t:

$$\begin{aligned} z\bar{z} - t^2 + 1 = 0 \quad &\Rightarrow \quad t^2 z_0\bar{z}_0 - t^2 + 1 = 0 \\ &\Rightarrow \quad t^2(z_0\bar{z}_0 - 1) = -1 \\ &\Rightarrow \quad t^2 = \frac{1}{1 - z_0\bar{z}_0} \\ &\Rightarrow \quad t = \pm\sqrt{\frac{1}{1 - z_0\bar{z}_0}} \end{aligned}$$

Since we want $t > 0$, we choose $t = \frac{1}{\sqrt{1-z_0\bar{z}_0}}$, which then gives us $z = \frac{z_0}{\sqrt{1-z_0\bar{z}_0}}$. This gives us the desired function $\psi : D^1 \to \{(z,t) : z\bar{z} - t^2 = -1,\ t > 0\}$ defined by

$$\psi(z) = \left(\frac{z}{\sqrt{1 - z\bar{z}}}, \frac{1}{\sqrt{1 - z\bar{z}}}\right).$$



Thus, we have the 3-dimensional Euclidean model of 2-dimensional Hyperbolic Geometry called the hyperboloid model. A point of interest with this model is that, with respect to the Minkowski metric $ds^2 = dx^2 + dy^2 - dt^2$, the surface we've projected onto is a sphere of radius i; Lambert had speculated that his acute hypothesis "described geometry on a 'sphere of imaginary radius'."[2]

6.5.2 History of the model

As we've said, the hyperboloid model of Hyperbolic Geometry dates back to Lambert's speculation that Hyperbolic Geometry could be modeled on a sphere of imaginary radius. However, this model was not thoroughly discussed or applied until the late 19^{th} century and early 20^{th} century. The first appearance of the hyperboloid model may have been in German mathematician Wilhelm Killing's piece on computations in Hyperbolic Geometry, "Die Rechnung in Nicht-Euclidischen Raumformen", published in Crelle's Journal in 1880. Killing attributed the ideas presented to Karl Weierstrass, and described the hyperboloid through Weierstrass coordinates.

In this same time period Poincaré was studying Fuchsian groups and quadratic forms. His work here led Poincaré to the same model Killing discussed in 1880, and to the conclusion that "the study of similarity substitutions of quadratic forms reduces to that of fuchsian groups" [4]. Poincaré subsequently published his work, "On the applications of noneuclidean geometry to the theory of quadratic forms" in 1881 and "Theory of Fuchsian Groups" in 1882 where, as Stillwell notes, the hyperboloid model is implicit.

Several years later the hyperboloid model was connected to the then-recent theory of special relativity presented by Einstein. Contributors to this connection, and the development of the hyperboloid as it relates to relativity, include Poincaré (1905-6), Herman Minkowski, and Jansen, in 1909, who Reynolds says gave "the first detailed exposition of H^2", the hyperboloid model of Hyperbolic Geometry. In the following chapter we will use this connection to further investigate H^2.

Chapter 7

Relativity, Minkowski Space, and the Hyperboloid H^2

"Subjects of our perception are always places and times connected. No one has observed a place except at a particular time, or has observed a time at a particular place. Yet I respect the dogma that time and space have independent existences each." Hermann Minkowski, 1909

7.1 Special Theory of Relativity

In 1905 Albert Einstein published his paper "On the Electrodynamics of Moving Bodies", which introduced the special theory of relativity. This theory, which utilizes contributions of Hendrik Lorentz and Henri Poincaré, connects time and space in a way contradictory to many assumptions held by physicists at the time of its introduction. Perhaps the most immediate of these stems from the work of James Clark Maxwell who, in 1865, published the now famous Maxwell equations. In his studies Maxwell discovered the existence of electromagnetic waves and that they travel at the speed of light, commonly denoted c. The measure of the speed of electromagnetic waves will never deviate, in a vacuum, regardless of the observer's state of motion. This result, which Einstein accepted and used in his special theory of relativity, appeared paradoxical because it implies that the distance traveled and time taken to travel will be measured differently by observers in different locations, or in different states of motion. Nonetheless, it is one of three "Postulates of Special Relativity" assumed by Einstein.

Central to the theory of relativity are the Einstein field equations, the solutions of which are metrics. In particular, we are concerned with the Minkowski metric, which may be given generally as

$$ds^2 = -dx_0^2 + \sum_{i=1}^{n} dx_i^2$$

and is a solution to the field equations of a vacuum; when $n = 2$, this is the metric we will associate with H^2, the hyperboloid model of Hyperbolic Geometry.

7.2 The Minkowski Metric

When the Minkowski metric given by $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ is associated with $\mathbb{R}^{1,3}$ we have a model of space-time (with this metric $\mathbb{R}^{1,3}$ is referred to as Minkowski space) for Einstein's special theory of relativity. While classic Minkowski space is $\mathbb{R}^{1,3}$, we can also associate $\mathbb{R}^{1,2}$ with

$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2$$

and get a model of space time[1]. This version of Minkowski space is where we will build our hyperboloid. Before we begin this construction, consider what's called a *light cone*.

[1]This is a model with only two spatial dimensions, whereas classic Minkowski space has three.

7.3 Light-Cones

Working in Minkowski space (and generally the theory of relativity) the speed of light is denoted by c; we'll let $c = 1$. If a flash of light were to occur at the origin, the path the light particles follow through space and time is given by what is called a light-cone; we can think of this as the set of points satisfying $x_1^2 + x_2^2 - x_0^2 = 0$.

Notice that we have a future light-cone ($x_0 > 0$) and a past light-cone ($x_0 < 0$), and the x_1y_2-plane (or simply $\mathbb{C}$) representing the present. Future (or past) events (points in $\mathbb{R}^{1,2}$) are separated by this light-cone into three types: light-like, time-like, and space-like. Light-like events are those events laying on the light-cone itself, and are causally affected by the event E of the light emission. Space-like events are those outside the light-cone ($x_1^2 + x_2^2 - x_0^2 > 0$); these are not causally affected by E. Time-like events are those inside the cone ($x_1^2 + x_2^2 - x_0^2 < 0$); these may be causally affected by E.

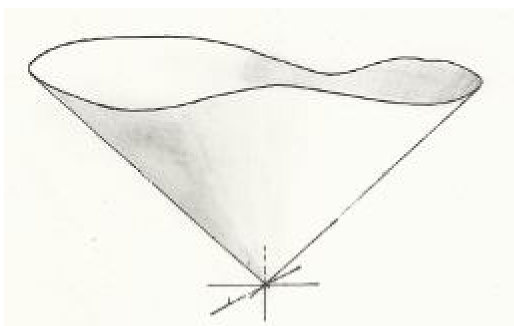


Figure 7.1: Future light cone

This light-cone will be particularly important when we begin discussing the notion of distance on our model.

7.4 The Hyperboloid, H^2

7.4.1 Reconstructing H^2

As we've said, we'll be using three-dimensional Minkowski space, denoted $\mathbb{R}^{1,2}$ or M^3, whose metric is given by $ds^2 = -dx_0^2 + dx_1^2 + dx_2^2$ where x_0 is the time coordinate, and x_1 and x_2 are spatial coordinates.

Our hyperboloid model of Hyperbolic Geometry, H^2, is set in M^3, and one approach to building and discussing this model begins with a Minkowski quadratic form,

$$q((x_0, x_1, x_2)) = -x_0^2 + x_1^2 + x_2^2 = [x_0\ x_1\ x_2] \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = X^t J_0 X.$$

Note that this can be generalized for a model of $(n-1)$-dimensional Hyperbolic Geometry in M^n by taking our quadratic form to be

$$q_n\left(\sum_{i=0}^{n} x_i U_i\right) = \sum_{i=0}^{n} e_i x_i^2,$$

where $\mathcal{U} = \{U_0, U_1, ..., U_n\}$ is a basis for M^n, $q_n(U_i) = e_i$, $e_0 = -1$, and $e_i = 1$ for $i \neq 0$. For much of this section we will, for the sake of brevity, take $\mathcal{U}$ to be a basis for M^3 and let $X = \sum_{i=0}^{2} U_i x_i$.

Along with this quadratic form, a bilinear form, p, will be useful in our development of the model.

Definition 7.1 A *bilinear form* on a vector space V is a function $f : V \times V \to \mathbb{F}$, where $\mathbb{F}$ is a field of scalars, such that

i. $f(u + v, w) = f(u, w) + f(v, w)$,

ii. $f(u, v + w) = f(u, v) + f(u, w)$, and

iii. $f(\lambda u, v) = \lambda f(u, v) = f(u, \lambda v)$.

For any $X, Y \in M^3$, define the bilinear form p by

$$p(X,Y) := \frac{1}{2}[q(X+Y) - q(X) - q(Y)] = X^t J_0 Y.$$

Notice that $q(X) = p(X, X)$, and that $p(U_i, U_j) = e_i$ if $i = j$, and $p(U_i, U_j) = 0$ if $i \neq j$. Furthermore, the hyperboloid on which we'll model Hyperbolic Geometry, discussed in chapter 6.5, is given by

$$q(X) = -1, \quad x_0 > 0. \tag{7.1}$$

For the Minkowski metric, 7.1 gives us the sphere with radius i, while $q(x) = 1$ gives us the sphere of radius 1 (a hyperboloid of one sheet in the Euclidean sense). With our model more clearly held within M^3, let us return briefly to the discussion of the theory of relativity and light cones.

The light-cone, as we've discussed it, acts as the asymptote for the (upper-half of a two-sheeted) hyperboloid described in 7.1[2]. Then, looking at the hyperboloid as an observer of flat space-time, H^2 is a circle whose radius increases faster than the speed of light.

Now, we've already seen some of the basic concepts needed for H^2 as a model of Hyperbolic Geometry; in the following sections we will more completely describe H^2 as a model, essentially putting the geometry onto H^2.

The ideas we have so far, and some new-ish ideas:

Lines on H^2 are the non-empty intersections of planes passing through the origin of M^3 with H^2.

[2]With the Minkowski metric, the light-cone is the separating barrier for the sphere of radius i and the sphere of radius 1, $q(x) = -1$ with $x_0 > 0$ and $q(x) = 1$.

Betweenness for points of a line should come naturally from the appearance of a line or, alternatively, how we initially constructed lines on H^2; projecting from the origin, through the Klein disk, onto H^2.

Separation of H^2, as the hyperbolic plane, by a line should also be clear; a line l will separate H^2 into two "half planes". As with previous models, we'll say that two points (or other geometric objects) are on opposite sides of l if one point lies on one of these half planes, and the second point lies on the second half-plane. From here the Plane Separation Postulate (our Neutral Geometry axiom 9) comes easily.

7.4.2 Distance

Reynolds [5] begins the discussion of distance from two points A and B of H^2 by looking at a partition of the segment $\overline{AB}$, $\mathcal{P} = \{P_0 = A,\ P_1,\ ...,\ P_n = B\}$, and then defines the distance between A and B to be

$$d(A, B) = \lim_{\|\mathcal{P}\| \to 0} \sum_{j=1}^{n} \sqrt{q(P_j - P_{j-1})},$$

where $\|\mathcal{P}\| = \max_{1 \leq j \leq n}\{d_e(P_j, P_{j-1})\}$ and $d_e(P_j, P_{j-1})$ is the Euclidean distance from P_j to P_{j-1}, so that $q(P_j - P_{j-1})$ is intended as a squared length. Since this would require each $q(P_j - P_{j-1})$ to be positive, the fact will have to be verified.

To argue that $q(P_j - P_{j-1}) > 0$, it's convenient to return to our discussion of relativity and mix in some Euclidean ideas. Recall that $q(X) = 0$ gives us our light-cone in M^3, and that $q(X) < 0$ for our time-like vectors inside the light-cone. So considering $P_j - P_{j-1}$ as a vector of M^3, we want to show that $P_j - P_{j-1}$ is space-like.

Now, since P_j and P_{j-1} are points of a differentiable $\mathbf{e} - path$ (our $\mathbf{h}$-line l), the mean value theorem tells us that there is some point W on $\mathbf{h}\overline{P_j P_{j-1}}$ such that a vector tangent to H^2 at W will be parallel to $P_j - P_{j-1}$. Recall that H^2 may be thought of as a circle whose radius increases faster than the speed of light, and that we let $c = 1$, which gives us that a light-like (sometimes called null) vector will have slope 1, time-like vectors will have slope greater than 1, and space-like vectors will have slope less than 1. Then the movement of a particle passing through the origin (or, relating back to the meaning of our coordinates, passing through the spatial origin at the present time, $x_0 = 0$) and moving, at a constant speed, faster than the speed of light in M^3 will be represented by a vector with slope less than one. Therefore, a vector tangent to H^2, in particular any vector tangent to H^2 at W, will have slope less than one, from which it follows that $P_j - P_{j-1}$ has a slope less than one. Since it is emanating from the origin, $P_j - P_{j-1}$ is therefore space like. Hence, $q(P_j - P_{j-1}) > 0$.

We can then use a parameterization of $\overline{AB}$ to evaluate this limit. Taking $F(v)$ be this parameterization, assume that $v \in [a, b]$ and $F(a) = A$ and $F(b) = B$. Associating this to the partition of $\overline{AB}$, we can then partition $[a, b]$ by $v_0 = a, v_1, ..., v_n = b$, so that $F(v_j) = P_j$. Clearly F should be differentiable, so take F' to be the derivative of F. Then we have

$$\begin{aligned}
d(A,B) &= \lim_{\|\mathcal{P}\|\to 0} \sum_{j=1}^{n} \sqrt{q(P_j - P_{j-1})} \\
&= \lim_{\|\mathcal{P}\|\to 0} \sum_{j=1}^{n} \sqrt{\frac{q(F(v_j) - F(v_{j-1}))}{(v_j - v_{j-1})^2}}\ (v_j - v_{j-1}) \\
&= \lim_{\|\mathcal{P}\|\to 0} \sum_{j=1}^{n} \sqrt{q\left(\frac{F(v_j) - F(v_{j-1})}{v_j - v_{j-1}}\right)}\ (v_j - v_{j-1}) \\
&= \int_a^b \sqrt{q(F'(v))}\ dv.
\end{aligned}$$

Now, before we can use this to give an explicit formula for distance in H^2, we will consider a special case and then discuss some transformations that will be helpful for fully developing a general distance formula.

The special case we'll consider is distance between two points on the line in H^2 described by $q(x_0, x_1, 0) = -1$, or equivalently $x_0 = \sqrt{x_1^2 + 1}$ with $x_2 = 0$.[3] Following [5], we'll call this line H^1 and take $A = a_0U_0 + a_1U_1$ and $B = b_0U_0 + b_1U_1$ to be points of H^1; then we can take $v \in [a_1, b_1]$ and define our parameterization by $F(v) = \sqrt{v^2+1}U_0 + vU_1$, so that $F'(v) = \frac{v}{\sqrt{v^2+1}}U_0 + U_1$. This gives us

$$\begin{aligned} d(A,B) &= \int_{a_1}^{b_1} \sqrt{q(F'(v))}\, dv \\ &= \int_{a_1}^{b_1} \sqrt{-\frac{v^2}{v^2+1} + 1}\, dv \\ &= \ln(v + \sqrt{v^2+1})\Big|_{a_1}^{b_1} \\ &= \operatorname{arcsinh}(b_1) - \operatorname{arcsinh}(a_1) \end{aligned}$$

In particular, if $A = U_0$, then $d(U_0, B) = \operatorname{arcsinh}(b_1)$ if $b_1 > 0$[4]. Then, if we let $r = \operatorname{arcsinh}(v)$, then $v = \frac{e^r - e^{-r}}{2} = \sinh(r)$ and, recalling from our parameterization, $x_0 = \sqrt{v^2+1} = \frac{e^r + e^{-r}}{2} = \cosh(r)$. This gives us a new parameterization of H^1,

$$P(r) = U_0 \cosh(r) + U_1 \sinh(r), \qquad r \in \mathbb{R},$$

which will become useful when we generalize our distance formula.

7.4.3 Orthogonal Transformations

Definition 7.2 Orthogonal Transformation: *A linear transformation $T : M^3 \to M^3$ is called an orthogonal transformation with respect to q if it preserves q.*

$T : M^3 \to M^3$ is orthogonal if $q(T(X)) = q(X)$.

---

[3]This is given by the intersection of the plane $x_2 = 0$ with H^2 or, relating back to $\mathfrak{K}$, this is the projection of the line $l = \{z = x + iy : y = 0\}$ in $\mathfrak{K}$ onto H^2 by ψ.

[4]For definitions and properties of the hyperbolic functions, see Appendix B.

Theorem 7.3 *Given a linear transformation $T : M^3 \to M^3$, T is an orthogonal transformation of M^3 if and only if $p(T(X), T(Y)) = p(X, Y)$ for each $X, Y \in M^3$ or, equivalently,*

$$[T]^t J_0 [T] = J_0$$

where $[T]^t$ is the transpose of matrix $[T]$ and J_0 is the metric tensor of M^3,

$$J_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Proof: Given a linear transformation $T : M^3 \to M^3$, we'll first assume that T is an orthogonal transformation of M^3; then for any $X \in M^3$ we know that $q(T(X)) = q(X)$. Then, from the definition of p, for any $X, Y \in M^3$

$$\begin{aligned} p(T(X), T(Y)) &= \frac{1}{2}\left(q(T(X) + T(Y)) - q(T(X)) - q(T(Y))\right) \\ &= \frac{1}{2}\left(q(T(X + Y)) - q(T(X)) - q(T(Y))\right) \\ &= \frac{1}{2}\left(q(X + Y) - q(X) - q(Y)\right) \\ &= p(X, Y). \end{aligned}$$

On the other hand, if we suppose that $p(T(X), T(Y)) = p(X, Y)$ for every $X, Y \in M^3$, then it follows immediately that

$$q(T(X)) = p(T(X), T(X)) = p(X, X) = q(X).$$

∎

Theorem 7.4 *The orthogonal transformations of M^3 form a group, called the orthogonal group $O(M^3)$.*

Proof: It's clear that the identity $T(X) = X$ is an orthogonal transformation of M^3. Since each orthogonal transformation preserves distance, if T is an orthogonal transformation of M^3 it is necessarily one-to-one and onto, so an inverse transformation exists, $T^{-1}(X) = [T]^{-1}X$. Now, if T is an orthogonal transformation of M^3 we want T^{-1} to also be an orthogonal transformation of M^3. We can express T^{-1} as follows,

$$T^t J_0 T = J_0 \Leftrightarrow J_0 T^t J_0 T = I \Leftrightarrow J_0 T^t J_0 = T^{-1}, \tag{7.2}$$

so that

$$(T^{-1})^t J_0 T^{-1} = (J_0 T^t J_0)^t J_0 (J_0 T^t J_0) = J_0 T J_0 J_0 (J_0 T^t J_0) = J_0 T (J_0 T^t J_0) = J_0 T T^{-1} = J_0,$$

and we have that T^{-1} is also an orthogonal transformation of M^3. Finally, if T and S are orthogonal transformations of M^3 then $(T \circ S)(X) = ([T][S])X$, and

$$([T][S])^t J_0([T][S]) = [S]^t[T]^t J_0[T][S] = [S]^t J_0[S] = J_0$$

by the previous theorem, so $O(M^3)$ is closed under composition, hence a group. ∎

Definition 7.5 It follows from Theorem 7.3 that if $T \in O(M^3)$ then T has determinant ± 1; since $T^t J_0 T = J_0$, using the fact that $\det(T^t) = \det(T)$ we get

$$-1 = \det(J_0) = \det(T^t J_0 T) = \det(T^t)\det(J_0)\det(T) = -\det(T)^2,$$

so $\det(T) = \pm 1$. The elements of $O(M^3)$ with determinant 1 form a group $O^+(M^3)$ called the *special orthogonal group*; this is a subgroup of $O(M^3)$ of index 2.

Returning to H^2, let $X \in H^2$ and $T \in O(M^3)$, so that $q(T(X)) = q(X) = -1$. It follows that either $T(X) \in H^2$ or $-T(x) \in H^2$; however, T is a linear transformation, and therefore continuous. Note that since T is continuous, the image of any connected set under T must also be connected. However, the upper- and lower-sheet of the two sheeted hyperboloid described by $q(X) = -1$ do not together form a connected set, nor does any union of nonempty subsets of each; in other words, if H and K are non-empty subsets of the upper- and lower-sheet respectively, then $H \cup K$ is not connected. Therefore, either $T(X) \in H^2$ for all $X \in H^2$, or $-T(X) \in H^2$ for all $X \in H^2$. Now, the collection of $T \in O(M^3)$ which send H^2 to itself form yet another group, $G(M^3)$. This admits another group, $G^+(M^3) = G(M^3) \cap O^+(M^3)$, which is a subgroup of $G(M^3)$ of index 2.

Remark 10 *This group, $G^+(M^3)$, is referred to as the 2-dimensional Lorentz group, often denoted $O(1,2)$. Recall that we're working in 3-dimensional space-time; the Lorentz group is usually discussed for classic Minkowski space, 4-dimensional space-time, and is denoted $O(1,3)$. This may be generalized, as we've done for 3-dimensions, to n-dimensional space-time (for $n \geq 2$) where this group would be called the $(n-1)$-dimensional Lorentz group and denoted $O(1, n-1)$.*

7.4.4 Generalizing Distance on H^2

Returning to the special case of distance on H^1, we'll restrict ourselves (briefly) to Minkowski 2-space, M^2. There are groups $O(M^2)$, $O^+(M^2)$, etc. analogous to those discussed in the previous section, and we'll use these here.

Theorem 7.6 *$T \in O(M^2)$ if and only if*

$$[T] = \begin{bmatrix} e\cosh(s) & f\sinh(s) \\ e\sinh(s) & f\cosh(s) \end{bmatrix}$$

where $e = \pm 1$, $f = \pm 1$, and s are uniquely determined by T.

Proof: As in Definition 7.5, because $T^t J_0 T = J_0$ and $J_0 = J_0^{-1}$, we know that

$$\det(T^t)\det(J_0)\det(T) = \det(J_0) = -1,$$

so $\det(T) = \pm 1$, and, as in line (7.2), we know

$$J_0 T^t J_0 T = I,$$

so it must be that $J_0 T^t J_0 = T^{-1}$. If

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then it follows that

$$T^{-1} = \pm \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

so that

$$J_0 T^t J_0 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & -c \\ -b & d \end{bmatrix} = \pm \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

This leads us to consider two cases:

1. if $\det T = 1$, then
$$\begin{bmatrix} a & -c \\ -b & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
which implies that $a = d$ and $b = c$,

2. and if $\det T = -1$ then
$$\begin{bmatrix} a & -c \\ -b & d \end{bmatrix} = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$$
which implies that $a = -d$ and $b = -c$.

For the first case we have that $ad - bc = a^2 - b^2 = 1$, and for the second we have $ad - bc = -a^2 + b^2 = -1$; each case leads us to $a = \pm\cosh(s)$ and $b = \pm\sinh(s)$ for some s, so that for the first case

$$T = \begin{bmatrix} \pm\cosh(s) & \pm\sinh(s) \\ \pm\sinh(s) & \pm\cosh(s) \end{bmatrix},$$

and for the second case

$$T = \begin{bmatrix} \pm\cosh(s) & \pm\sinh(s) \\ \mp\sinh(s) & \mp\cosh(s) \end{bmatrix}$$

or

$$T = \begin{bmatrix} \pm\cosh(s) & \mp\sinh(s) \\ \pm\sinh(s) & \mp\cosh(s) \end{bmatrix}.$$

∎

Corollary 7.4.1 *Given e and f from Theorem 7.6,*

i. $T \in G(M^2)$ *if and only if* $e = 1$*,*

ii. $T \in O^+(M^2)$ *if and only if* $e = f$*, and*

iii. $T \in G^+(M^2)$ *if and only if* $e = f = 1$*.*

Proof:

i. Let T be in $O(M^2)$, and consider $X = \begin{bmatrix} 1, 0 \end{bmatrix}$.

$$\begin{aligned} TX &= \begin{bmatrix} e\cosh(s) & f\sinh(s) \\ e\sinh(s) & f\cosh(s) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} e\cosh(s) \\ e\sinh(s) \end{bmatrix} \end{aligned}$$

Since $G(M^2)$ is defined to be those elements of $O(M^2)$ which map H^1 to itself, $T \in G(M^2)$ if and only if $e\cosh(s) > 0$, so $T \in G(M^2)$ if and only if $e = 1$.

ii. Recall that $O^+(M^2)$ is the subgroup of $O(M^2)$ containing all elements of $O(M^2)$ with determinant 1. As we saw in the proof of Theorem 7.6, if $[T]$ has determinant 1 then $e = f$, and if $[T]$ has determinant -1 then $e = -f$.

iii. $T \in G^+(M^2) = G(M^2) \cap O^+(M^2)$ if and only if $e = 1$, by part i., and $e = f$, by part ii., so $T \in G^+(M^2) = G(M^2) \cap O^+(M^2)$ if and only if $e = f = 1$.

∎

Now, returning to M^3, define a new subgroup G_1 of $G(M^3)$ where $T \in G_1$ if and only if T map H^1 to itself.

Theorem 7.7 *If $T \in G_1$ then T fixes M^2, and using*

$$T = \begin{bmatrix} t_{0,0} & t_{1,0} & t_{2,0} \\ t_{0,1} & t_{1,1} & t_{2,1} \\ t_{0,2} & t_{1,2} & t_{2,2} \end{bmatrix},$$

$t_{2,0} = t_{2,1} = 0$, from which it follows that $t_{0,2} = t_{1,2} = 0$; so $T \in G_1$ will fix the subspace spanned by U_2.

Proof: Let $T \in G_1$, and X lie on H^1, so that

$$TX = \begin{bmatrix} t_{0,0} & t_{1,0} & t_{2,0} \\ t_{0,1} & t_{1,1} & t_{2,1} \\ t_{0,2} & t_{1,2} & t_{2,2} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_0 t_{0,0} + x_1 t_{1,0} \\ x_0 t_{0,1} + x_1 t_{1,1} \\ x_0 t_{0,2} + x_1 t_{1,2} \end{bmatrix} = Y$$

is also on H^1. Then $x_0 t_{0,2} + x_1 t_{1,2} = 0$ for all X on H^1; this includes $X = \begin{bmatrix} 1, 0, 0 \end{bmatrix}$, so it must be that $t_{0,2} = 0$. Then $x_1 t_{1,2} = 0$ for all X on H^1, which implies that $t_{1,2}$ is also 0.

Now, since $J_0T^tJ_0 = T^{-1}$ (from line (7.2)), we have

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_{0,0} & t_{0,1} & 0 \\ t_{1,0} & t_{1,1} & 0 \\ t_{2,0} & t_{2,1} & t_{2,2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} t_{0,0} & -t_{0,1} & 0 \\ -t_{1,0} & t_{1,1} & 0 \\ -t_{2,0} & t_{2,1} & t_{2,2} \end{bmatrix} = T^{-1},$$

but T^{-1} must also map H^1 to itself, so $t_{2,0} = t_{2,1} = 0$.

So for any $T \in G_1$ and $X = [x_0, x_1, 0]$ we have

$$TX = \begin{bmatrix} t_{0,0} & t_{1,0} & 0 \\ t_{0,1} & t_{1,1} & 0 \\ 0 & 0 & t_{2,2} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_0t_{0,0} + x_1t_{1,0} \\ x_0t_{0,1} + x_1t_{1,1} \\ 0 \end{bmatrix}.$$

So if $T \in G_1$ then T maps M^2 back to M^2, and the subspace spanned by U_2 back to itself. Also, since $T^tJ_0T = J_0$ it must be that $t_{2,2} = \pm 1$; notice that

$$\begin{aligned} T^tJ_0T &= \begin{bmatrix} t_{0,0} & t_{0,1} & 0 \\ t_{1,0} & t_{1,1} & 0 \\ 0 & 0 & t_{2,2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_{0,0} & t_{1,0} & 0 \\ t_{0,1} & t_{1,1} & 0 \\ 0 & 0 & t_{2,2} \end{bmatrix} \\ &= \begin{bmatrix} -t_{0,0}^2 - t_{0,1}^2 & -t_{0,0}t_{1,0} - t_{0,1}t_{1,1} & 0 \\ t_{0,0}t_{1,0} + t_{0,1}t_{1,1} & t_{0,0}^2 + t_{0,1}^2 & 0 \\ 0 & 0 & t_{2,2}^2 \end{bmatrix} = J_0, \end{aligned}$$

so $t_{2,2} = \pm 1$. Combining this with Theorem 7.6 we have that if $T \in G_1$ then

$$T = L_sJ_1^iJ_2^j$$

where $i, j = 0, 1$ are exponents,

$$L_s = \begin{bmatrix} \cosh(s) & \sinh(s) & 0 \\ \sinh(s) & \cosh(s) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for some real s, and

$$J_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

$\blacksquare$

From here, we have that $L_s L_t = L_{s+t}$ and $L_0 = I$, the identity transformation of M^3:

$$\begin{aligned} L_s L_t &= \begin{bmatrix} \cosh(s) & \sinh(s) & 0 \\ \sinh(s) & \cosh(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh(t) & \sinh(t) & 0 \\ \sinh(t) & \cosh(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cosh(s)\cosh(t) + \sinh(s)\sinh(t) & \cosh(s)\sinh(t) + \sinh(s)\cosh(t) & 0 \\ \sinh(s)\cosh(t) + \cosh(s)\sinh(t) & \sinh(s)\sinh(t) + \cosh(s)\cosh(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cosh(s+t) & \sinh(s+t) & 0 \\ \sinh(s+t) & \cosh(s+t) & 0 \\ 0 & 0 & 1 \end{bmatrix} = L_{s+t} \end{aligned}$$

Similarly, defining G_0 to be the subgroup of $G(M^3)$ consisting of transformations which fix U_0, we have that $T \in G_0$ will fix the subspace spanned by U_1 and U_2; since $T \in G_0$ fixes U_0, its inverse also fixes U_0, and so T is of the form

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & t_{1,1} & t_{1,2} \\ 0 & t_{2,1} & t_{2,2} \end{bmatrix}.$$

However,because $q(x_1U_1 + x_2U_2) = x_1^2 + x_2^2$, this subspace spanned by U_1 and U_2 is a Euclidean plane with the restriction of q giving the usual Euclidean metric. Now, if T is an orthogonal transformation of a Euclidean plane then it is a rotation or reflection, and has the form

$$T = \begin{bmatrix} \cos\theta & -h\sin\theta \\ \sin\theta & h\cos\theta \end{bmatrix}$$

for $h = \pm 1$, and T would be in the special orthogonal group of the Euclidean plane if and only if $h = 1$ [5]. So, if $T \in G_0$, then

$$T = R_\theta J_2^j,$$

again with $j = 0, 1$, for some θ where R_θ is a transformation of the subspace spanned by U_1 and U_2 by rotation of θ,

$$R_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

Theorem 7.8 $G_1 \cap G_0 = \{I, R_{180^\circ}, J_1, J_2\}$.

Proof: If $T \in G_0 \cap G_1$, then

$$T = L_s J_1^i J_2^j = R_\theta J_2^k$$

for some s and θ, $i, j, k = 0, 1$. This forces $\cosh(s) = 1$, which implies that $s = 0$. Our options for T then are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$$

where $a, b = \pm 1$; these elements form the set $\{I, R_{180^\circ}, J_1, J_2\}$, each of which is in G_0 and G_1.

∎

Now, recall the parameterization P we gave for H^1. If we apply R_θ to this parameterization we get

$$P(r, \theta) = R_\theta(P(r)) = U_0 \cosh r + U_1 \cos\theta \sinh r + U_2 \sin\theta \sinh r;$$

this is a parameterization of H^2; recall that this is a sphere with imaginary radius in M^3, and notice the similarity between this parameterization and that of S^2:

$$(\sin(\psi)\cos(\theta), \sin(\psi)\sin(\theta), \cos(\psi)).$$

Using this parameterization we can now define hyperbolic translations and rotations; using matrix multiplication and properties of the hyperbolic functions, $L_s(P(r, 0)) = P(r + s, 0)$ and $R_\theta(P(r, \phi)) = P(r, \phi + \theta)$, so that L_s is a translation by s along H^1 and R_θ is a rotation about U_0 by θ.

Now, returning to a direct discussion of distance on H^2, let A and B be points of H^2 as before, with l the line passing through A and B. Applying the appropriate transformations from G we may map A to U_0 and B to some point B' of H^1, $B' = P(r, 0)$. However, recall that elements of G preserve the bilinear form p, so it follows that

$$p(A, B) = p(U_0, B') = p(P(0, 0), P(r, 0)) = p((1, 0, 0), (\cosh r, \sinh r, 0)) = -\cosh r.$$

However, r is the distance from U_0 to B', which is the distance from A to B, so we have

$$r = d(A, B) = \operatorname{arccosh}(-p(A, B)).$$

7.5 Lines

Recall that lines in H^2 are to be the intersection of planes passing through the origin with the upper-half of the two sheeted hyperboloid $q(X) = -1$. Since three points determine a unique plane, it follows that through two points of H^2 there exists a unique line. Furthermore, the equation of any line in H^2 will be given by the equation of the plane which defines the line,

$$p(V, X) = -v_0x_0 + v_1x_1 + v_2x_2 = 0,$$

for some non-zero $V \in M^3$. If X is on this line then $q(X) = -1$, so we can assume that $q(V) = 1$, so that V lies on the hyperboloid of one sheet (the sphere of radius 1 with respect to the Minkowski metric); denote this subset of M^3 by D^2,

$$D^2 := \{V \in M^3 : q(V) = 1\}.$$

Now, notice that, for some fixed $V \in D^2$, if X satisfies $p(V, X)$ then X also satisfies $p(-V, X)$; let $\pm V$ be the *poles* of the line $l = \{X \in H^2 : p(V, X) = 0\}$.

7.6 Isometries

This group G is the group of isometries of H^2. To show this, we'll be using the following theorem.

Theorem 7.9 *Let l and m be lines on H^2, $\overrightarrow{AB}$ a ray of l and $\overrightarrow{CD}$ a ray of m, and R a side of l and S a side of m. There exists exactly one $T \in G$ such that $T(A) = C$, $T(l) = m$, $T(\overrightarrow{AB}) = \overrightarrow{CD}$, and $T(R) = S$.*

Proof: Recall from our discussion generalizing distance that a ray may be mapped to a ray of the line H^1 by some element of G, and in fact that any ray may be mapped so to the ray of H^1 emanating from U_0 such that $x_1 \geq 0$. Then each of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ may each be mapped to this ray of H^1 by some T_1 and T_2 in G, where $T_1(A) = U_0$ and $T_2(C) = U_0$. It follows that l and m are then mapped to H^1 by T_1 and T_2 respectively. If T_1 maps R to the same side of H^1 to which S is mapped by T_2, then $T_2^{-1} \circ T_1$ is the desired element of G; letting $T = T_2^{-1} \circ T_1$, we have that $T(A) = C$, $T(l) = m$, $T(\overrightarrow{AB}) = \overrightarrow{CD}$, and $T(R) = S$. If, on the other hand, R and S are mapped to different sides of H^1 by T_1 and T_2, then J_2 will map $T_1(R)$ to $T_2(S)$ and fix $T_1(l) = T_2(m) = H^1$, so that $T_2^{-1} \circ J_2 \circ T_1$ is the desired element of G, so that for $T = T_2^{-1} \circ J_2 \circ T_1$ we have $T(A) = C$, $T(l) = m$, $T(\overrightarrow{AB}) = \overrightarrow{CD}$, and $T(R) = S$.

∎

Now, an isometry of H^2 is a function $\rho : H^2 \to H^2$ which preserves distance, so by the previous theorem, restricting elements of G to H^2 results in an isometry of H^2. However, our claim is that every isometry of H^2 is given by some element of G. To show that this is true we'll use the above theorem and the fact that an isometry is determined by what it does to three non-collinear points. Suppose that ρ is an isometry of H^2, and consider a triangle ΔABC on H^2. Letting ρ map A, B, and C to A', B', and C' respectively, because ρ is and isometry $\Delta ABC \cong \Delta A'B'C'$ by the side-side-side congruence condition, Theorem 2.29. However, by Theorem 7.9 there is some $T \in G$ which maps A to A' and $\overrightarrow{AB}$ to $\overrightarrow{A'B'}$; since T is an isometry, we know that $\overline{A'B'} \cong \overline{AB} \cong \overline{A'T(B)}$, implying that $T(B) = B'$. By the same theorem we may assume that T maps C to the same side of $\overleftrightarrow{A'B''}$ as C'. It then follows that $T(C) = C'$, so $\rho = T|_{H^2}$, so that every isometry of H^2 is an fact the restriction of some element of G.

7.7 Trigonometry

Definition 7.10 *Suppose that* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *are rays in* H^2 *such that* $\overleftrightarrow{AB} \neq \overleftrightarrow{AC}$*. If* V *and* W *are vectors in* M^3 *tangent to* $\overline{AB}$ *and* $\overline{AC}$ *respectively at* A *such that* $q(V) = q(W) = 1$*, then the measure of the angle* $\angle BAC$ *is given by*

$$m\angle BAC = \arccos(p(V, W)).$$

Consider a triangle ΔABC in H^2, and take $d_{\mathbb{H}}(A, B) = c$, $d_{\mathbb{H}}(C, B) = a$, $d_{\mathbb{H}}(A, C) = b$, and $m\angle ABC = \beta$, , $\angle BCA = \gamma$, $m\angle CAB = \alpha$. We know that there exists a T in G which would map C to U_0, A to the ray of H^1 for which $x_1 \geq 0$, and B to the side of H^1 for which $x_2 > 0$, and because such T is an isometry of H^2 ΔABC and its image under T would be congruent; that the side lengths of the two triangle are congruent is clear. It follows that corresponding angles of these two triangles are congruent, but to give this assertion more credibility, consider the following: if V and W are vectors tangent to $\overline{AB}$ and $\overline{AC}$ respectively, then $T(V)$ and $T(W)$ are also vectors tangent to $\overline{T(A)T(B)}$ and $\overline{T(A)T(C)}$ respectively. Also, because $T \in G$, T preserves the bilinear form p, in other words, $q(T(V)) = p(T(V), T(V)) = p(V, V) = 1$, $q(T(W)) = p(T(W), T(W)) = p(W, W) = 1$, and $p(V, W) = p(T(V), T(W))$, so T preserves angles. Due to this fact, we can assume that ΔABC is already such a triangle; let $C = U_0$, A be a point of the ray of H^1 for which $x_1 \geq 0$, and let B lie on the side of H^1 for which $x_2 > 0$. Then, using our parameterization

$$P(r, \theta) = R_\theta(P(r)) = U_0 \cosh r + U_1 \cos\theta \sinh r + U_2 \sin\theta \sinh r,$$

we have $A = P(b, 0)$ and $B = P(a, \gamma)$. Now, recall that L_s is a translation by $-b$ along H^1, so that L_{-b} will map $A = P(b, 0)$ to $A' = P(0, 0) = U_0$. Similarly, L_{-b} will map $C = U_0$ to $C' = P(-b, 0) = P(b, 180°)$; essentially L_{-b} is sliding C along

H^1 for $x_1 \leq 0$. As for $B = P(a, \gamma)$, applying L_{-b} will also translate B to the point $B' = P(c, 180° - \alpha)$; this last comes from the fact that our parameterization gives us a point X on on H^2 in terms of its distance from U_0 and the angle formed by $\overrightarrow{U_0X}$ and H^1 for $x_1 \geq 0$. Since the distance from B' to $A' = U_0$ is c, and the angle formed by $\overrightarrow{U_0B'}$ and H^1 with $x_1 \leq 0$ is α, it follows that $B' = P(c, 180° - \alpha)$.

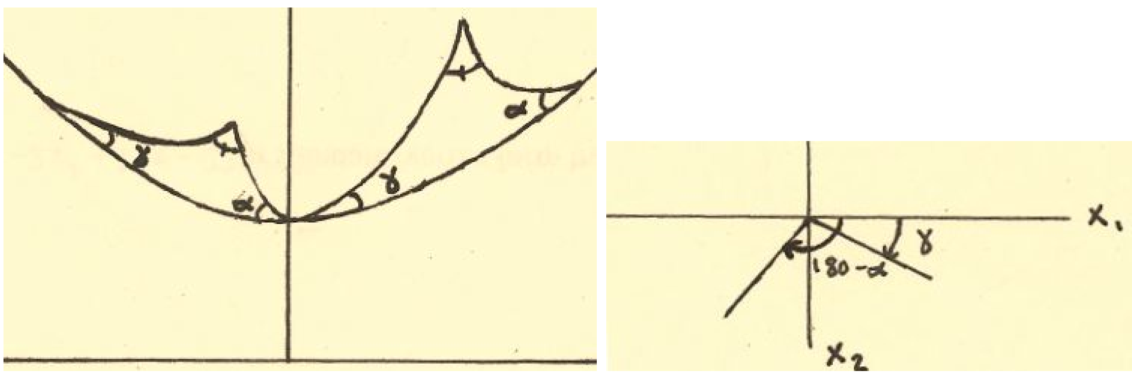


Figure 7.2: Translating ΔABC along H^1.

In figure 7.2 we have, on the left, the triangle ΔABC and its image $\Delta A'B'C'$. On the right, we have the plane tangent to H^2 at U_0, illustrating our identification of B with $P(a, \gamma)$ and B' with $P(c, 180° - \alpha)$.

Then, looking a little more closely at the equation $B' = L_{-b}(B)$, we have

$$\begin{bmatrix} \cosh(c) \\ -\sinh(c)\cos(\alpha) \\ \sinh(c)\sin(\alpha) \end{bmatrix} = \begin{bmatrix} \cosh(b) & -\sinh(b) & 0 \\ -\sinh(b) & \cosh(b) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh(a) \\ \sinh(a)\cos(\gamma) \\ \sinh(a)\sin(\gamma) \end{bmatrix}.$$

Simplifying the right side of this equation yields

$$\begin{bmatrix} \cosh(c) \\ -\sinh(c)\cos(\alpha) \\ \sinh(c)\sin(\alpha) \end{bmatrix} = \begin{bmatrix} \cosh(b)\cosh(a) - \sinh(b)\sinh(a)\cos(\gamma) \\ -\sinh(b)\cosh(a) + \cosh(b)\sinh(a)\cos(\gamma) \\ \sinh(a)\sin(\gamma) \end{bmatrix},$$

which gives us our next theorem.

Theorem 7.11 *Given a triangle* ΔABC*, take* $d_{\mathbb{H}}(A,B) = c$*,* $d_{\mathbb{H}}(C,B) = a$*,* $d_{\mathbb{H}}(A,C) = b$*, and* $m\angle ABC = \beta$*, ,*$\angle BCA = \gamma$*,* $m\angle CAB = \alpha$*. Then the following hold:*

1. *The hyperbolic law of cosines,*

$$\cos(\gamma) = \frac{\cosh(a)\cosh(b) - \cosh(c)}{\sinh(a)\sinh(b)}$$

2. *The hyperbolic law of sines,*

$$\frac{\sin(\alpha)}{\sinh(a)} = \frac{\sin(\gamma)}{\sinh(c)} = \frac{\sin(\beta)}{\sinh(b)}.$$

From the hyperbolic law of cosines we get the following corollary, analogous to Pythagoras' Theorem for Euclidean Geometry.

Corollary 7.7.1 *If* ΔABC *is a right triangle, with* $m\angle ABC = 90°$*, then letting* $d_{\mathbb{H}}(A,B) = c$*,* $d_{\mathbb{H}}(A,C) = b$*, and* $d_{\mathbb{H}}(B,C) = a$*, the side lengths of* ΔABC *are related as follows;*

$$\cosh(b) = \cosh(a)\cosh(c).$$

7.8 Circles

Unlike $\mathbb{H}$, $\mathfrak{P}$, and $\mathfrak{H}$, the conformal models of Hyperbolic Geometry we've seen, in H^2 hyperbolic circles will not necessarily appear as euclidean circles. Circles in H^2 are given by the intersection of a plane, not passing through the origin or tangent to H^2, with H^2, from which it follows that three points will determine a unique circle. In general, the equation of a circle is that of the plane which defines it,

$$p(V, X) = -v_0x_0 + v_1x_1 + v_2x_2 = -k$$

for some fixed $V \in M^3 \setminus \{\mathbf{0}\}$ and $k \in \mathbb{R} \setminus \{0\}$. This is the first theorem we see in this section.

Theorem 7.12 *Each circle γ lies on a plane given by*

$$p(V, X) = -v_0x_0 + v_1x_1 + v_2x_2 = -k$$

for some fixed $V \in M^3 \setminus \{\mathbf{0}\}$ and $k \in \mathbb{R} \setminus \{0\}$

As with much of our previous work, we can prove this theorem by first considering a relatively simple case. Here, we'll begin with a circle C whose center is U_0, so that for each X on C, $X = P(r, \theta)$ for $\theta \in [0, 360)$ where r is the radius of the circle. This circle is clearly the intersection of a plane which is parallel (in the euclidean sense) to the x_1x_2-plane. Notice that X lies on this circle satisfies

$$P(U_0, X) = -\cosh r.$$

Let's denote by S the plane whose intersection with H^2 is C. Now, if we have a second circle C' with center $V = P(d, \theta)$ and radius r, then we can map C to C' by applying a translation followed by a rotation; applying L_d to U_0 will map U_0 to $P(d, 0)$, and following this up with the rotation R_θ results in V, $R_\theta L_d(U_0) = V$. Since $R_\theta L_d$ is an isometry, each point X of C will be mapped to a point X' such that the distance from X' to V is r, so X' lies on C'; hence, the image of C under $R_\theta L_d$ is C'.Furthermore, since $T = R_\theta L_d$ preserves the bilinear form p, we have that for $T(X) = X'$

$$p(V, X') = p(U_0, X) = -\cosh r.$$

Finally, this isometry T clearly maps S to a second plane S'; any element of G will map two-dimensional subspaces of M^3 to one another, and T specifically is a rotation about the x_0-axis and translation about the path H^1. Then we have

$$T(C) = T(S \cap H^2) = S' \cap H^2 = C'.$$

Chapter 8

Fuchsian Groups: Returning to $\mathbb{H}$

In this chapter we will be revisiting our first model of Hyperbolic Geometry, $\mathbb{H}$. We'll be investigating isometries and the topology of $\mathbb{H}$, and what are known as Fuchsian groups. Before delving back into the half plane model, we'll need some definitions and terminology for later discussion.

8.1 Topology, Bundles, Group Action, Etc.

Definition 8.1 Let X be a metric space and G be a group of homeomorphisms of X. Then a family $\{M_\alpha : \alpha \in A\}$ of subsets of X indexed by elements of a set A is called *locally finite* if for any compact subset $K \subset X$, $M_\alpha \cap K \neq \emptyset$ for only finitely many $\alpha \in A$. If the orbit of any point $x \in X$ is locally finite then G acts *properly discontinuously* on X.

Lemma 8.1.1 *If a group G acts properly discontinuously on a set X then each orbit Gx is discrete and each stabilizer G_x is finite.*

Proof: Supposing that G acts properly discontinuously on X, we know that each orbit Gx is locally finite. If there is some Gx' which is not discrete, then every neighborhood U of x' there are infinitely many $g \in Gx'$ such that $g \cdot x' \in U$; this would contradict our assumption that Gx' is locally finite, so each orbit must be discrete. Since each Gx is discrete, each $x \in X$ has a neighborhood U such that the set of $g \in G$ mapping x into U is finite; such a subset of G contains G_x, so G_x must also be finite. ∎

Theorem 8.2 *A group G acts properly discontinuously on a set X if and only if each $x \in X$ has a neighborhood V such that*

$$g(V) \cap V \neq \emptyset \; for \; only \; finitely \; many \; g \in G, \tag{8.1}$$

where $g(V) = \{g \cdot v : v \in V\}$.

Proof: First suppose that G acts properly discontinuously on X. Then each orbit Gx is discrete and for each $x \in X$, G_x is finite. Therefore, there exists a ball centered at x with radius ϵ, $B(x, \epsilon)$, which contains no point of Gx other than x. Then for any neighborhood V of x contained in $B(x, \epsilon)$, $g(V) \cap V \neq \emptyset$ implies that $g(x) = x$, so $g \in G_x$. Since G_x is finite, there exist only finitely many $g \in G$ such that $g(V) \cap V \neq \emptyset$.

We'll argue the other direction by contrapositive. If there exists Gx which is not discrete, then there exists an accumulation point x_0 of Gx, so that every neighborhood of x_0 contains infinitely many points of Gx; each neighborhood of x_0 will contain infinitely many points of its image under G. So if Gx is not discrete then (8.1) does not hold. Similarly, if $g \cdot x = x$ for infinitely many $g \in G$, then $g(V) \cap V \neq \emptyset$ for infinitely many g, and (8.1) does not hold. So, if (8.1) does hold Gx must be discrete and, for each $x \in X$, G_x is finite; hence, G acts properly discontinuously on X.

∎

8.2 Isometries of the Half-Plane

Definition 8.3 Given a geometry (K, d) and function $f : (K, d) \to (K, d)$ where $d(A, B)$ is the distance from A to B for all $A, B \in K$, if $d(A, B) = d(f(A), f(B))$ for all $A, B \in K$, then f is called an *isometry* of (K, d).

Here we'll specifically discuss isometries of $\mathbb{H}$ (denoting the half-plane model of Hyperbolic Geometry). We'll see here that many möbius transformations are isometries of $\mathbb{H}$ (clearly not all möbius transformations are isometries since we used the transformation $z \to \frac{-z+i}{zi-1}$ to send $\mathbb{H}$ to $\mathbb{D}$). We'll also discuss Isom($\mathbb{H}$), the group of all isometries on $\mathbb{H}$ (we'll need to show that this is in fact a group), and we'll discuss the idea of "orientation-preserving".

Consider the group of 2×2 matrices with determinant 1,

$$SL_2(\mathbb{R}) = \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, \det(g) = 1 \right\}.$$

This group is called the unimodular group, and is directly related to the group of *real* linear fractional (möbius) transformations

$$\mathbb{M}_{\mathbb{R}} = \left\{ z \to \frac{az+b}{cx+d} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

just as $\mathbb{M}$ is related to $\mathrm{GL}_2(\mathbb{C})$;

$$\mathbb{M}_{\mathbb{R}} \cong \mathrm{PSL}_2(\mathbb{R}) = \frac{\mathrm{SL}_2(\mathbb{R})}{\pm I_2}.$$

Theorem 8.4 $\mathrm{PSL}_2(\mathbb{R})$ *acts on* $\mathbb{H}$ *by homeomorphisms.*

Proof: For z in $\mathbb{H}$ and g in $\mathrm{PSL}_2(\mathbb{R})$, $g \cdot z = \frac{az+b}{cz+b}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R}),$$

so $g \cdot z = m(z)$ for some $m \in \mathbb{M}_{\mathbb{R}}$. We want $m|_{\mathbb{H}}$ to be a homeomorphism on $\mathbb{H}$. Since $m \in \mathbb{M}_{\mathbb{R}}$ m^{-1} exists, and m and m^{-1} are both continuous. To show that m is a homeomorphism, we also need that m is a bijection on $\mathbb{H}$; however, if m maps $\mathbb{H}$ to $\mathbb{H}$ for any $m \in \mathbb{M}_{\mathbb{R}}$ then this is also true of m^{-1}, and m would be a bijection on $\mathbb{H}$. So we need to ensure that m will map $\mathbb{H}$ to $\mathbb{H}$; let $m \in \mathbb{M}_{\mathbb{R}}$ and $z \in \mathbb{H}$, and consider $m(z) = w$. Then we have

$$\begin{aligned} w = m(z) &= \frac{az+b}{cz+d} \\ &= \frac{(az+b)(a\bar{z}+d)}{|cz+d|^2} \\ &= \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz+d|^2} \end{aligned}$$

Then the imaginary part of w, $\Im(w)$, is given by

$$\Im(w) = \frac{ad\Im(z) + bc\Im(\bar{z})}{|cz+d|^2} = \frac{ad\Im(z) - bc\Im(z)}{|cz+d|^2} = \frac{(ad-bc)\Im(z)}{|cz+d|^2} = \frac{\Im(z)}{|cz+d|^2}.$$

Then $\Im(z) > 0$ implies that $\Im(w) > 0$. Hence m is a bijection on $\mathbb{H}$, and is therefore a homeomorphism on $\mathbb{H}$. ∎

Theorem 8.5 $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{Isom}(\mathbb{H})$.

Proof: Recall from Theorem 4.39 that möbius transformations preserve the ratio $(AB, PQ) = \frac{|AQ||BP|}{|AP||BQ|}$, and that distance from A to B in $\mathbb{H}$ is defined to be $d_{\mathbb{H}} = |ln((AB, PQ))|$ where P and Q are ideal points of $\overleftrightarrow{AB}$; note that one of P

and Q may be the point at infinity in $\bar{\mathbb{C}}$ if $\overleftrightarrow{AB}$ appears as a euclidean ray. Since $PSL_2(\mathbb{R})$ acts on $\mathbb{H}$ by homeomorphisms, specifically elements of $\mathbb{M}(\mathbb{R})$, elements of $PSL_2(\mathbb{R})$ also preserve the ratio (AB, PQ), and therefore preserve distance. Hence, $PSL_2(\mathbb{R}) \subseteq \mathrm{Isom}(\mathbb{H})$. For the strict containment, note that reflection across the imaginary axis, $z \to -\bar{z}$, is also an isometry of $\mathbb{H}$ since it clearly preserves (AB, PQ). However, this is not a möbius transformation, so no element of $\mathrm{PSL}_2(\mathbb{R})$ may be represented by $z \to -\bar{z}$, and we have $PSL_2(\mathbb{R}) \subset \mathrm{Isom}(\mathbb{H})$.

∎

Lemma 8.2.1 *For any line l in $\mathbb{H}$, there exists an element g of* $\mathrm{PSL}_2(\mathbb{R})$ *such that g maps l to the imaginary axis.*

Proof: We have two cases to consider: when l appears as a euclidean ray, and when l appears as a Euclidean semicircle. When l appears as a ray, we have

$$l = \{z = x + yi : x = a\}$$

for some real a. Clearly $m(z) = z - a$ will map l to the imaginary axis, so we want $m \in \mathbb{M}_{\mathbb{R}}$ or, equivalently,

$$g = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R}).$$

Clearly, $\det(g) = 1$, so the result holds in the first case.

For our second case, our line l will be given by

$$l = \left\{ z = x + yi : \left| z - \left(\frac{a+b}{2} \right) \right|^2 = \left| \frac{b-a}{2} \right|^2 \right\}$$

where a and b are the ideal points of l. We're searching for a transformation which will map one ideal point to 0, and the second to ∞; choosing $m(z) = \frac{\lambda(z-b)}{\lambda(z-a)}$ will satisfy this requirement. Then we need the appropriate λ such that

$$g = \begin{bmatrix} \lambda & -\lambda b \\ \lambda & -\lambda a \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R}).$$

Setting $\det(g) = -\lambda^2 a + \lambda^2 b = 1$ gives $\lambda = \frac{1}{\sqrt{b-a}}$.

∎

Theorem 8.6 *If $z, w \in \mathbb{H}$, then*

(i) $d_{\mathbb{H}}(z,w) = \ln \frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|}$

(ii) $\cosh d_{\mathbb{H}}(z,w) = 1 + \frac{|z-w|^2}{2\Im(z)\Im(w)}$

(iii) $\sinh(\frac{1}{2}d_{\mathbb{H}}(z,w)) = \frac{|z-w|}{2(\Im(z)\Im(w))^{1/2}}$

(iv) $\cosh(\frac{1}{2}d_{\mathbb{H}}(z,w)) = \frac{|z-\bar{w}|}{2(\Im(z)\Im(w))^{1/2}}$

(v) $\tanh(\frac{1}{2}d_{\mathbb{H}}(z,w)) = \left|\frac{z-w}{z-\bar{w}}\right|$.

Proof: Since (ii)-(v) all follow from (i) [6], it is sufficient to prove (i)[1]. This proof takes on two sides, and to begin we'll consider the case where $A = z$ and $B = w$ are points in $\mathbb{H}$ such that $\overleftrightarrow{AB}$ appears as a euclidean ray. As we know from chapter 5, specifically from line 5.1 in our discussion of axiom 2, the distance from between these points is then

$$d_{\mathbb{H}}(z,w) = |\ln((AB, PQ))| = \left| ln \frac{\Im(z)}{\Im(w)} \right|,$$

where P and Q are the ideal points of $\overleftrightarrow{AB}$. Since $\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|} \geq 1$, we can assume wlog that $\Im(z) \geq \Im(w)$[2], so that what we have to show is that

$$\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|} = \frac{\Im(z)}{\Im(w)}.$$

[1]The computations for $(ii) - (iv)$ can be found in Appendix B.

[2]The argument does not change if, instead, $\Im(z) \leq \Im(w)$; we would have $\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|z-w} = \frac{\Im w}{\Im z} = e^{d_{\mathbb{H}}(z,w)}$ and our conclusion is the same.

By assumption $\Re(z) = \Re(w)$, so $|z-w| = \Im(z-w)$ and $|z-w| = \Im(z-\bar{w})$. Then $\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|}$ becomes

$$\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|} = \frac{\Im(z-\bar{w})+\Im(z-w)}{\Im(z-\bar{w})-\Im(z-w)} = \frac{2\Im(z)}{2\Im(w)} = \frac{\Im(z)}{\Im(w)} = e^{d_{\mathbb{H}}(z,w)},$$

so that $d_{\mathbb{H}}(z,w) = \ln\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|}$.

Now, for second side to this proof we're concerned with when $\overleftrightarrow{AB}$ appears as a semicircle. However, if we can show that $\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|}$ is preserved under action by $g \in \mathrm{PSL}_2(\mathbb{R})$, then the proof would be complete since $d_{\mathbb{H}}(x,w)$ is preserved under action by g by Theorem 8.5. Let g_0 be the element of $\mathrm{PSL}_2(\mathbb{R})$ which maps $\overleftrightarrow{AB}$ to the imaginary axis (see Lemma 8.2.1), so that $\Re(g_0 \cdot z) = \Re(g_0 \cdot w) = 0$. Then, letting m_0 be the element of $\mathbb{M}_{\mathbb{R}}$ corresponding to g_0,

$$\frac{|m_0(z)-\overline{m_0(w)}|+|m_0(z)-m_0(w)|}{|m_0(z)-\overline{m_0(w)}|-|m_0(z)-m_0(w)|} = \frac{\left|\frac{az+b}{cz+d}-\frac{a\bar{w}+b}{c\bar{w}+d}\right|+\left|\frac{az+b}{cz+d}-\frac{aw+b}{cw+d}\right|}{\left|\frac{az+b}{cz+d}-\frac{a\bar{w}+b}{c\bar{w}+d}\right|-\left|\frac{az+b}{cz+d}-\frac{aw+b}{cw+d}\right|} \tag{8.2}$$

For now, consider $\left|\frac{az+b}{cz+d}-\frac{a\bar{w}+b}{c\bar{w}+d}\right|$:

$$\begin{aligned}\left|\frac{az+b}{cz+d}-\frac{a\bar{w}+b}{c\bar{w}+d}\right| &= \left|\frac{(acz\bar{w}+bc\bar{w}+adz+bd)-(acz\bar{w}+bcz+ad\bar{w}+bd)}{(cz+d)(c\bar{w}+d)}\right| \\ &= \left|\frac{z(ad-bc)-\bar{w}(ac-bd)}{(cz+d)(c\bar{w}+d)}\right| \\ &= \left|\frac{z-\bar{w}}{(cz+d)(c\bar{w}+d)}\right|\end{aligned}$$

Applying this to (8.2), we have

$$\begin{aligned}\frac{|m_0(z)-\overline{m_0(w)}|+|m_0(z)-m_0(w)|}{|m_0(z)-\overline{m_0(w)}|-|m_0(z)-m_0(w)|} &= \frac{\left|\frac{z-\bar{w}}{(cz+d)(c\bar{w}+d)}\right|+\left|\frac{z-w}{(cz+d)(cw+d)}\right|}{\left|\frac{z-\bar{w}}{(cz+d)(c\bar{w}+d)}\right|-\left|\frac{z-w}{(cz+d)(cw+d)}\right|} \\ &= \frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|}\end{aligned}$$

since $|(cz+d)(c\bar{w}+d)| = |(cz+d)(cw+d)|$. Therefore, $d_{\mathbb{H}}(z,w) = \ln\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|}$ holds for all points $A = z$ and $B = w$ if it holds for points for which the line containing them appears as a ray. Having already shown this holds in such cases, the proof is complete. ∎

Definition 8.7 *Define* $S^*L_2(\mathbb{R}) := \{g \in \mathrm{GL}_2(\mathbb{R}) : \det(g) = \pm 1\}$ *so that* $\mathrm{SL}_2(\mathbb{R})$ *is a subgroup of* $S^*L_2(\mathbb{R})$ *of index two;*

$$S^*L_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) \rtimes \langle K \rangle ,$$

where $\langle K \rangle$ *is congruent to the cyclic group of order two,* C_2, *and*

$$K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Further define

$$PS^*L_2(\mathbb{R}) := \frac{S^*L_2(\mathbb{R})}{\pm I_2},$$

so that $\mathrm{PSL}_2(\mathbb{R})$ *is a subgroup of* $PS^*L_2(\mathbb{R})$ *with index two;*

$$PS^*L_2(\mathbb{R}) = \mathrm{PSL}_2(\mathbb{R}) \rtimes \langle K \rangle .$$

Remark 11 *Euclidean dilation centered at the origin and with ratio* $k > 0$ *is an isometry of* $\mathbb{H}$ *represented in* $\mathbb{M}_{\mathbb{R}}$ *by* $m(z) = kz$ *and in* $\mathrm{PSL}_2(\mathbb{R})$ *by*

$$g = \begin{bmatrix} \frac{k}{\sqrt{k}} & 0 \\ 0 & \frac{1}{\sqrt{k}} \end{bmatrix}.$$

Remark 12 *Given distinct points* A, B, *and* C, $d_{\mathbb{H}}(A, C) = d_{\mathbb{H}}(A, B) + d_{\mathbb{H}}(B, C)$ *if and only if* $A - B - C$.

Lemma 8.2.2 *If* ρ *is an isometry of* $\mathbb{H}$ *then* ρ *sends lines in* $\mathbb{H}$ *to lines in* $\mathbb{H}$.

Proof: If ρ is an isometry of $\mathbb{H}$, then $d_{\mathbb{H}}(\rho(z), \rho(w)) = d_{\mathbb{H}}(z, w)$ for each $z, w \in \mathbb{H}$. Take A, B, and C to be distinct collinear points such that $A - B - C$.

Suppose that ρ maps A, B, and C to A', B', and C' respectively. Then it follows from remark 12 that

$$d_{\mathbb{H}}(A,C) = d_{\mathbb{H}}(A,B) + d_{\mathbb{H}}(B,C) = d_{\mathbb{H}}(A',B') + d_{\mathbb{H}}(B',C').$$

However, ρ is an isometry, so $d_{\mathbb{H}}(A,C) = d_{\mathbb{H}}(A',C')$, so it must be that $A' - B' - C'$, and ρ maps $\overleftrightarrow{AC}$ to $\overleftrightarrow{A'B'}$. ∎

Lemma 8.2.3 *If an isometry ρ maps the imaginary axis I to itself and fixes i, then either $\rho(z) = z$ or $\rho(z) = -\frac{1}{z}$ for each z on I.*

Proof: Assuming that ρ maps I to itself and fixes i, consider some z on I distinct from i. If $\rho(z) = w$, then we know that w is on I and

$$d_{\mathbb{H}}(i,z) = d_{\mathbb{H}}(i,w)$$

since ρ is an isometry which fixes i. However, there are only two such points on I, one of which is z. To see that $-\frac{1}{z}$ is the second, consider the following; if z lies on I then $z = ki$ for some $k > 0$ and $-\frac{1}{z} = \frac{i}{k}$, so $-\frac{1}{z}$ also lies on I, and

$$\begin{aligned} e^{d_{\mathbb{H}}\left(i,-\frac{1}{z}\right)} &= \frac{\left|i + \frac{i}{k}\right| + \left|i - \frac{i}{k}\right|}{\left|i + \frac{i}{k}\right| - \left|i - \frac{i}{k}\right|} \\ &= \frac{\frac{1}{k}\left|ki + i\right| + \frac{1}{k}\left|ki - i\right|}{\frac{1}{k}\left|ki + i\right| - \frac{1}{k}\left|ki - i\right|} \\ &= \frac{|ki - \bar{i}| + |ki - i|}{|ki - \bar{i}| - |ki - i|} \\ &= e^{d_{\mathbb{H}}(z,i)}. \end{aligned}$$

Therefore, if ρ fixes i and maps I to itself, then either ρ fixes I or $\rho(z) = -\frac{1}{z}$ for each z on I. ∎

Theorem 8.8 $\mathrm{Isom}(\mathbb{H})$ *is generated by the möbius transformations in* $\mathrm{PSL}_2(\mathbb{R})$ *and the mapping $z \to -\bar{z}$, and is isomorphic to $PS^*L_2(\mathbb{R})$.*

Proof: By Lemma 8.2.2, if ρ is an isometry then it maps lines to lines, and so maps the imaginary axis I to some line $\rho(I)$. By Lemma 8.2.1, there exists $g \in \mathrm{PSL}_2(\mathbb{R})$ (and associated $m \in \mathbb{M}_{\mathbb{R}}$) such that $g \cdot \rho(I) = m(\rho(I))$ is the imaginary axis. By applying the isometry $z \to kz$, for some $k > 0$, we may assume that i is fixed by $m \circ \rho$. Then by Lemma 8.2.3 either $m \circ \rho$ fixes I or $(m \circ \rho)(z) = -\frac{1}{z}$; so, if necessary, we can apply the isometry $z \to -\frac{1}{z}$ to be sure that all of I is fixed by $m \circ \rho$.

Now, take $z = x + yi$ not on I and $m(\rho(z)) = w = u + vi$, so that for all $t > 0$ we have

$$d_{\mathbb{H}}(z, it) = d_{\mathbb{H}}(w, it).$$

Then it follows from Theorem 8.6(iii) that

$$\frac{|z - it|}{2\sqrt{\Im(z)\Im(it)}} = \frac{|w - it|}{2\sqrt{\Im(w)\Im(it)}} \Rightarrow [x^2 + (y - t)^2]v = [u^2 + (v - t)^2]y$$

for all $t > 0$. Then dividing both sides of the second equation above by t^2 and letting t tends towards infinity yields

$$\lim_{t \to \infty} \frac{[x^2 + (y - t)^2]v}{t^2} = v$$

and

$$\lim_{t \to \infty} \frac{[u^2 + (v - t)^2]y}{t^2} = y,$$

so $v = y$, and therefore $x^2 = u^2$. This implies that $w = z$ or $w = -\bar{z}$. If $w = z$, consider a third point $m(\rho(r)) = r' = s' + t'i$; we require $d_{\mathbb{H}}(z, r) = d_{\mathbb{H}}(z, r')$, and if $r' = \bar{r}$ then these distance are not equal since z is not on I, so it must be that $r' = r$. Therefore, if $w = z$ this holds for all z and $m \circ \rho$ is the identity, so $\rho = m^{-1}$, and is therefore associated to some element of $\mathrm{PSL}_2(\mathbb{R})$. If $w = -\bar{z}$, then

$$\rho(z) = m^{-1}(-\bar{z}) = \frac{-d\bar{z} - b}{c\bar{z} - a}$$

where $m(z) = \frac{az+b}{cz+d}$. Notice that $(-d)(-a) - (c)(-b) = bc - ad = -1$, so ρ is not in $\mathbb{M}_{\mathbb{R}}$ (or associated to any element of $\mathrm{PSL}_2(\mathbb{R})$). Thus any isometry of $\mathbb{H}$ is generated by $z \to -\bar{z}$ and elements of $\mathbb{M}_{\mathbb{R}}$. ∎

Definition 8.9 Let the trace of $g \in \mathrm{PSL}_2(\mathbb{R})$ be denote $\mathrm{Tr}(g)$. If a transformation $g \in \mathrm{PSL}_2(\mathbb{R})$ is not the identity, it is called *elliptic* if $(\mathrm{Tr}(g))^2 < 4$, *parabolic* if $(\mathrm{Tr}(g))^2 = 4$, and *hyperbolic* if $(\mathrm{Tr}(g))^2 > 4$.

This terminology is in reference to the curves in $\mathbb{R}^2$ invariant under the action of the matrices [6]. A matrix in $\mathrm{SL}_2(\mathbb{R})$, which is not the identity, is called hyperbolic when it is diagonalizable over $\mathbb{R}$, or is conjugate to a unique matrix

$$\begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}$$

in $\mathrm{SL}_2(\mathbb{R})$. The invariant curves of such a matrix's action on $\mathbb{R}^2$ are hyperbolas, hence the term hyperbolic transformation. Similarly, an elliptic transformation is given by a matrix conjugate in $\mathrm{SL}_2(\mathbb{R})$ to a unique matrix of the form

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

and the invariant curves of such a transformation are ellipses. Referring to the remaining transformations as parabolic is analogous to parabolas being the curves intermediate between hyperbolas and ellipses.

Theorem 8.10 *A hyperbolic transformation has two fixed points in* $\mathbb{R} \cup \{\infty\}$

Proof: Assuming that g is a hyperbolic transformation, we want to solve

$$z = \frac{az+b}{cz+d}.$$

This leads us to $cz^2 + dz - az - b = 0$, from which we have

$$\begin{aligned} z &= \frac{(a-d) \pm \sqrt{(d-a)^2 + 4cb}}{2c} \\ &= \frac{(a-d) \pm \sqrt{(d^2 - 2ad + a^2 + 4cb}}{2c} \\ &= \frac{(a-d) \pm \sqrt{(d+d)^2 - 4(ad-cb)}}{2c} \\ &= \frac{(a-d) \pm \sqrt{(\mathrm{Tr}(g))^2 - 4}}{2c}. \end{aligned}$$

Since $(\mathrm{Tr}(g))^2 > 4$, there are two real solutions, and so g fixes two points of $\mathbb{R} \cup \{\infty\}$. In particular, if $c = 0$, so that

$$g = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix},$$

then one of these fixed points is ∞, and the other is $\frac{b}{d-a}$. ∎

Definition 8.11 A line in $\mathbb{H}$ joining two fixed points of a hyperbolic transformation g is the *axis of* g, denoted $C(g)$.

Notice that the fixed points of a hyperbolic transformation are points of $L \cup \{\infty\}$, so the fixed points of a hyperbolic transformation are ideal points. Then if g is hyperbolic fixing z_0 and ∞, then $C(g) = \{z \in \mathbb{H} : \Re(z) = z_0\}$, and if g fixes z_0 and z_1 then

$$C(g) = \left\{ z \in \mathbb{H} : \left| z - \frac{z_0 + z_1}{2} \right|^2 = \left| \frac{z_0 - z_1}{2} \right|^2 \right\}.$$

Theorem 8.12 *If g is hyperbolic then g maps $C(T)$ onto itself.*

Proof: Suppose that g is hyperbolic, so that g fixes two points z_0 and z_1; one of these fixed points may be ∞. Since g maps lines of $\mathbb{H}$ to lines of $\mathbb{H}$ (Lemma 8.2.2), the line whose ideal points are z_0 and z_1 must be mapped to itself. ∎

8.3 What is a Fuchsian Group?

Definition 8.13 A discrete subgroup of Isom($\mathbb{H}$) is called a *Fuchsian group* if it consists of orientation preserving transformations. So a Fuchsian group is a discrete subgroup of PSL$(2,\mathbb{R})$. Equivalently, a subgroup of Isom($\mathbb{H}$) is called a Fuchsian group if it acts properly discontinuously on $\mathbb{H}$.

Theorem 8.14 *In definition 8.9 we classified elements of* $\mathrm{PSL}_2(\mathbb{R})$ *as hyperbolic, parabolic, or elliptic.*

i. All hyperbolic and parabolic cyclic subgroups of $\mathrm{PSL}_2(\mathbb{R})$ *are Fuchsian groups.*

ii. An elliptic cyclic subgroup of $\mathrm{PSL}_2(\mathbb{R})$ *is a Fuchsian group if and only if it is finite.*

Our calling these groups Fuchsian comes from Poincaré labeling a type of periodic function, which are invariant under substitutions of the form

$$z \to \frac{az+b}{cz+d}, \quad ad - bc \neq 0,$$

as Fuchsian after the mathematician Lazarus Fuchs who discovered a large class of such functions. These functions and their groups originated from certain differential equations [4]. In studying these Fuchsian groups by means of tessellations, Poicané realized that linear fractional transformations could be used to define length in Hyperbolic (or Bolyai-Lobachevsky) Geometry. Later, in a paper published in 1881, Poincaré discussed the use of Hyperbolic Geometry in the study of quadratic forms, and came to the conclusion that "the study of similarity substitutions of quadratic forms reduces to that of fuchsian groups." This idea arose as Poincaré was studying curvilinear polygons on one half of a two sheeted hyperboloid, brought about by application of similarity substitutions which preserve the indefinite quadratic form describing the hyperboloid. By projecting this half-hyperboloid into the interior of a circle, Poincaré returned to what we've called the poincaré disk where the polygons remained polygons, but became bounded by circular arcs which we know to be lines in this model; this is where Poincaré had been studying Fuchsian groups.

Now, if Γ is a Fuchsian group, then, relating back to the polygons, there exists some closed region F of $\mathbb{H}$ (or $\mathfrak{P}$, or H^2, etc.) such that the action of Γ on F tessellates the hyperbolic plane; this region F is called the *fundamental domain* of Γ. In this way the study of Fuchsian groups leads us to the study of tessellations of the hyperbolic plane. From here, one is lead to the study of orbifolds, or orbit manifolds, which are given by the quotient $\Gamma\backslash\mathbb{H}$ where Γ is a Fuchsian group.

Appendix A

Definitions

Definition A.1 Given a line l and point A, l *contains* A if A lies on l. A is also said to be a point of l.

Definition A.2 Three points A, B, and C are non-collinear if they are not collinear: if there does not exist a line containing A, B, and C.

Definition A.3 Two lines *intersect* if there is a point which lies on both lines. More generally, two objects (circles, polygons, planes, etc.) *intersect* if a point lies on both objects.

Definition A.4 A set is *convex* if, for all points A and B contained by the set, the segment $\overline{AB}$ does not contain any point not contained by the set.

Definition A.5 Given a line l and distinct points A and B not on l, A and B are said to be on the *same side* of l when $\overline{AB}$ does not intersect l, and to be on *opposite sides* of l when $\overline{AB}$ does intersect l.

This gives us the convex sets mentioned in axiom 9; defining a set H to contain points which are on the same side of l, this set is clearly convex by the previous definition. We can similarly define a set K to contain points, not in H, which are on the same side of l:

$$H := \{A, B : \overline{AB} \cap l = \emptyset\}, \quad \text{and} \quad K := \{C, D : C, D \notin H, \overline{CD} \cap l = \emptyset\}.$$

It follows that if $A \in H$ and $B \in K$, then $\overline{AB}$ intersects l, so A and B are on opposite sides of l.

Definition A.6 Given two points A and B, and the unique line l passing through them, the *segment* $\overline{AB}$ is the set of all points of l lying between A and B, as well as A and B. $\overline{AB} = \{C \in l : A - C - B \text{ or } C = A \text{ or } C = B\}$.

Definition A.7 Given a triangle ΔABC, the *interior* of this triangle is the set of all points which are one the same side of $\overleftrightarrow{AB}$, on the same side of $\overleftrightarrow{BC}$, and on the same side of $\overleftrightarrow{AC}$.

Definition A.8 (i) A *right angle* is an angle whose measure is $90°$, (ii) an *acute angle* is an angle whose measure is less than $90°$, and (iii) an *obtuse angle* is an angle whose measure is greater than $90°$. Note that by Axiom 12 we measure angles only between $0°$ and $180°$.

Definition A.9 Two angles who share vertices and one side are *adjacent* if the second side of either angle is not interior to the other angle.

Definition A.10 Any angle that is both supplememtary and adjacent to an angle of a triangle is called an *exterior angle* of the triangle.

Definition A.11 Given two lines MN and OP who intersect at a point R such that $M - R - N$ and $O - R - P$, angles $\angle MRO$ and $\angle PRN$ are *vertical angles*, and $\angle ORN$ and $\angle PRM$ are *vertical angles*. More specifically, if angle A is adjacent and supplementary to both angles B and C, then B and C are vertical angles.

Definition A.12 Given a segment $\overline{AB}$, a point C is said to be the *midpoint* of $\overline{AB}$ if A, B, and C are collinear, and if $d(A, C) = d(C, B)$.

Definition A.13 Given angle $\angle ABC$, $\overrightarrow{BD}$ is an *angle bisector* of $\angle ABC$ if D is in the interior of $\angle ABC$ and $m(\angle ABD) = m(\angle DBC) = \frac{1}{2}m(\angle ABC)$.

Definition A.14 Lines l and m are said to be *perpendicular* if they intersect, and the angles formed by this intersection are right angles.

Definition A.15 A *circle with center P* is the set of all points equidistant from the point P; this distance is called the *radius* of the circle.

Definition A.16 A quadrilateral $\square ABCD$ is called a *Saccheri quadrilateral* if $\angle DAB$ and $\angle ABC$ are right angles, and $\overline{AD} \cong \overline{BC}$.

Definition A.17 A quadrilateral $\square ABCD$ is called a *rectangle* if each of its interior angles are right angles and opposite sides are congruent; $\overline{AD} \cong \overline{BC}$, $\overline{AB} \cong \overline{CD}$, and $m\angle ABC = m\angle BCD = m\angle CDA = m\angle DAB = 90°$.

Definition A.18 Given a Saccheri quadrilateral $\square ABCD$ such that $\angle DAB$ and $\angle ABC$ are right angles and $\overline{AD} \cong \overline{BC}$, $\overline{AB}$ is the *base* of $\square ABCD$ and $\overline{CD}$ is the *summit.*

Definition A.19 A quadrilateral $\square ABCD$ is called a *Lambert quadrilateral* if $\angle CDA$, $\angle DAB$, and $\angle ABC$ are right angles.

Definition A.20 Given triangles ΔABC and ΔXYZ, if each angle of ΔABC is congruent to a corresponding angle of ΔXYZ, then these triangles are *similar*, denoted $\Delta ABC \sim \Delta XYZ$. If follows that pairs of congruent triangles are also similar triangles.

Appendix B

Construction stuff that doesn't seem to fit anywhere else...

Throughout this section we'll be referencing different models of Hyperbolic Geometry. Recalling the notation we've used for each model, we have the half plane model $\mathbb{H}$, the Poincaré disk $\mathfrak{P}$, the hemisphere model $\mathfrak{H}$, the Klein disk $\mathfrak{K}$, and the hyperboloid model H^2.

B.1 The Klein Model $\mathfrak{K}$

Theorem B.1 *Let l be a line in $\mathfrak{K}$, and let l' be the corresponding line in $\mathfrak{P}$. If A' is on l', so that $\phi(A') = A$ is on l, then the euclidean ray $\overrightarrow{OA'}$ intersects l at A.*

The function ϕ in this theorem is the same function we used in the construction of $\mathfrak{K}$,

$$\phi(z) = \frac{2z}{|z|^2 + 1}.$$

It's easy to see that $A' = z$, $A\phi(z)$, and the origin O are collinear, in the euclidean sense, since the euclidean line passing through z and $\phi(z)$ is given by

$$tz + (1-t)\phi(z),$$

so that when $t = \frac{-2}{|z|^2-1}$ we get $tz+(1-t)\phi(z) = 0$. In fact, we can say that $0-z-\phi(z)$, again in the euclidean sense, so that the euclidean ray $\overrightarrow{OA'}$ passes through A. Of course, this theorem is also clear if one were to review our construction of $\mathfrak{K}$, where we used the inverse of stereographic projection to move from $\mathfrak{P}$ to $\mathfrak{H}$, and then projected $\mathfrak{H}$ back onto the unit disk to obtain $\mathfrak{K}$.

While we're relating back to the construction of $\mathfrak{K}$, recall that $\mathfrak{P}$ is conformal; because stereographic projection is a conformal mapping we also have that $\mathfrak{H}$ is also a conformal model of Hyperbolic Geometry. However, the Klein disk is *not* a conformal model, as our next theorem illustrates.

Theorem B.2 *Let l and m be lines in $\mathfrak{K}$. Then l is perpendicular to m if*

i. m appears as a diameter and $l \perp m$ in the euclidean sense, or

ii. if, when extended, l passes through the pole of m.

Proof: Let m be a line in $\mathfrak{K}$, and take P to be the pole of m. If T_1 and T_2 are the ideal points of m, then the (euclidean) circle C with center P and radius $\overline{PT_2}$ is orthogonal to the boundary of $\mathfrak{K}$; let's denote this by $\partial\mathfrak{K}$. This means that inversion through C will map $\partial\mathfrak{K}$ to itself, and if R is any point on $\partial\mathfrak{K}$ between T_1 and T_2 then $I_C(R) = S$, where S is the point on $\mathfrak{K}$ such that $P-R-S$. Recall from Theorem 4.24 that any circle passing through R and S will be orthogonal to C, so the lines in $\mathfrak{P}$ corresponding to $\overline{T_1T_2}$ and $\overline{RS}$ are perpendicular; hence the line l in $\mathfrak{K}$ with R and S as ideal points is perpendicular to m.

Notice that, because the ideal points of a line in $\mathfrak{P}$ will be interchanged by inverting through a circle (which gives another line in $\mathfrak{P}$ perpendicular to the first), this proof gives us that two lines in $\mathfrak{K}$, which do not both appear as diameters, are perpendicular if and only if, when extended, one line passes through the pole of the other. However, this does not quite cover the possibility that both l and m appear as diameters. That l and m, both appearing as diameters, are perpendicular if and only if they are perpendicular in the euclidean sense comes from the fact that $\mathfrak{P}$ is conformal and that our mapping ϕ from $\mathfrak{P}$ to $\mathfrak{K}$ fixes only the origin and ∂K. Because the origin is fixed $\mathfrak{K}$ is conformal at the origin (in fact $\mathfrak{K}$ is conformal *only* at the origin), and we have that l and m are perpendicular if and only if they are perpendicular in the euclidean sense.

∎

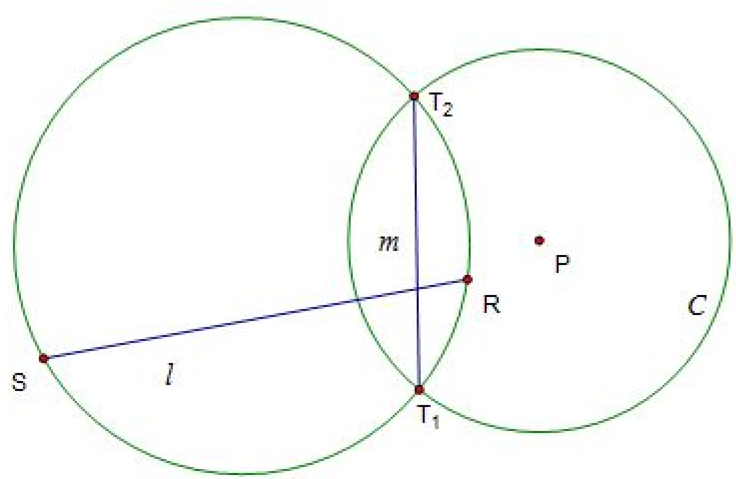


Figure B.1: Perpendicular lines in the Klein disk

Denote the ratio we've frequently used, $\frac{|AC||BD|}{|AD||BC|}$, by (AB, CD).

Definition B.3 If A, B, C, and D are distinct collinear points in the extended complex plane, then C and D are *harmonic conjugates* with respect to the segment $\overline{AB}$ (likewise, A and B are harmonic conjugates with respect to $\overline{CD}$) when $(AB, CD) = 1$. The segment joining all four points, with endpoints A, B, C, or D (so if $A - C - B - D$ then the segment we're concerned with is $\overline{AD}$), is called a *harmonic tetrad*; we will denote this segment by $\overline{ABCD}$.

Theorem B.4 *Given a segment $\overline{AB}$, if C is the midpoint of $\overline{AB}$ then the harmonic conjugate of C with respect to $\overline{AB}$ is the point at infinity.*

Proof: With the given situation, we're looking for a point D such that

$$(AB, CD) = \frac{|AC||BD|}{|BC||AD|} = 1.$$

Since $\frac{|AC|}{|BC|} = 1$, we also need $\frac{|BD|}{|AD|} = 1$. There are only two possible points, C and the point at infinity, but C and D must be distinct in order to be harmonic conjugates, so D must be the point at infinity. ∎

Theorem B.5 *If C and D are harmonic conjugates with respect to $\overline{AB}$, then either $A - C - B$ or $A - D - B$.*

Proof: Suppose that C and D are harmonic conjugates with respect to $\overline{AB}$, and that $A - B - D$. Then we have that

$$(AB, CD) = \frac{|AC||BD|}{|BC||AD|} = \frac{|AC||BD|}{|BC|(|AB| + |BD|)} = 1,$$

so that $\frac{|BC|}{|AC|} = \frac{|BD|}{|AB|+|BD|} < 1$. This means that the distance from A to C is greater than the distance from B to C, so $|AC| = |AB| \pm |BC|$. If it were that $|AC| = |AB| + |BC|$, then

$$1 = \frac{(|AB| + |BC|)|BD|}{|BC|(|AB| + |BD|)} \quad \Rightarrow \quad (|AB| + |BC|)|BD| = |BC|(|AB| + |BD|)$$
$$\Rightarrow \quad \frac{|AB|}{|BC|} = \frac{|AB|}{|BD|}$$

This implies that $C = D$, which contradicts the definition of harmonic conjugates. So $|AC| = |AB| - |BC|$, which implies that $A - C - B$. ∎

Now, suppose we are given that C and D are harmonic conjugates with respect to $\overline{AB}$, and let γ be the (euclidean) circle whose diameter is $\overline{AB}$. Then the midpoint M of $\overline{AB}$ is the center of γ, and $|AB| = 2r$ where r is the radius of γ. Now, since C and D are harmonic conjugates with respect to $\overline{AB}$ we know that $(AB, CD) = 1$, so

$$|AC||BD| = |BC||AD|.$$

However, notice that when we take $d_1 = |MC|$ and $d_2 = |MD|$ this leads us to

$$\begin{aligned} |AC||BD| = |BC||AD| \quad &\Leftrightarrow \quad (r + d_1)(d_2 - r) = (r - d_1)(r + d_2) \\ &\Leftrightarrow \quad r^2 = d_1 d_2. \end{aligned}$$

Thus C and D are mapped to one another under inversion through γ.

Corollary B.1.1 *If C and D are harmonic conjugates with respect to $\overline{AB}$, and γ is the circle with diameter $\overline{AB}$, then I_γ maps C to D and visa-versa.*

Definition B.6 Given line l and a point P not on l, $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is the *harmonic homology* with center P and axis l if:

1. $P \xrightarrow{h} P$,
2. if A lies on l, then $A \xrightarrow{h} A$,
3. and if A does not lie on l and is distinct from P, then $A \xrightarrow{h} A'$, where A' is the harmonic conjugate of A with respect to $\overline{MP}$ and M is the point at which $\overleftrightarrow{PA}$ intersects l; if the euclidean lines l and $\overleftrightarrow{PA}$ are parallel, then M is the point at infinity and A' is the point such that P is the midpoint of $\overline{AA'}$.

Definition B.7 Given lines l and m, and a point P not on l or m, a *perspectivity* with center P maps l to m such that, if A is on l, then the perspectivity with center P maps A to A', the point of intersection of $\overleftrightarrow{PA}$ with n; if $\overleftrightarrow{PA}$ and n are parallel, then A' is the point at infinity on n.

Given a triangle ΔABC, let $m\angle ABC = \beta$, $m\angle CAB = \alpha$, and $m\angle BCA = \gamma$. The law of sines states that

$$\frac{|AB|}{\sin\gamma} = \frac{|AC|}{\sin\beta} = \frac{|BC|}{\sin\alpha}.$$

Theorem B.8 *A perspectivity preserves the cross-ratio* (AB, CD) *of four collinear points* A*,* B*,* C*, and* D*.*

Proof: Consider distinct lines l and n, and a point P not on l or n, and let A, B, C, and D be distinct points of l. Then we want to show that

$$(AB, CD) = (A'B', C'D').$$

We have several triangles here, including ΔAPC and ΔPBC. It follows from the law of sines that

$$\frac{|AC|}{|BC|} = \frac{\sin(m\angle APC)}{\sin(m\angle BPC)}$$

and

$$\frac{|BD|}{|AD|} = \frac{\sin(m\angle BPD)}{\sin(m\angle APD)}.$$

Now, since $\angle APC \cong \angle A'PC'$, $\angle BPC \cong \angle B'PC'$, $\angle APD \cong \angle A'PD'$, and $\angle BPD \cong \angle B'PD'$, we have

$$
\begin{aligned}
(AB, CD) &= \frac{|AC|}{|BC|}\frac{|BD|}{|AD|} \\
&= \frac{\sin(m\angle APC)}{\sin(m\angle BPC)}\frac{\sin(m\angle BPD)}{\sin(m\angle APD)} \\
&= \frac{\sin(m\angle APC)}{\sin(m\angle BPC)}\frac{\sin(m\angle BPD)}{\sin(m\angle APD)} \\
&= \frac{\sin(m\angle A'PC')}{\sin(m\angle B'PC')}\frac{\sin(m\angle B'PD')}{\sin(m\angle A'PD')} \\
&= \frac{|A'C'|}{|B'C'|}\frac{|B'D'|}{|A'D'|} \\
&= (A'B', C'D').
\end{aligned}
$$

∎

Now, we want a distance function $d_{\mathfrak{K}}$ for our model $\mathfrak{K}$ so that, for points A and B of $\mathfrak{P}$ and corresponding points A' and B' in $\mathfrak{K}$, $d_{\mathfrak{K}}(A', B') = d_{\mathfrak{P}}(A, B)$. At this point we have several tools to make quick work of this problem. To start, we know that we can map lines in $\mathfrak{P}$ to a diameter of the unit disk, through a circle inversion (hyperbolic reflection in $\mathfrak{P}$), without changing the value of the cross-ratio (AB, PQ), where A and B are points of $\mathfrak{P}$ and P and Q are the ideal points of $\overleftrightarrow{AB}$. Also, by the previous theorem, we know that the same may be done with points A' and B' of $\mathfrak{K}$ through a perspectivity; notice that since A and A', and B and B', are the same points (represented in different models), P and Q are ideal points for both $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$. Then wlog we can assume that $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ appear as diameters. Since rotation also does not change the cross-ratio, we can assume that $P = -1$ and $Q = 1$, and take $A = z$ and $B = w$ (each of these are points on the real axis of the complex plane).

With all of this in mind, we will once again use the cross ratio to help define distance. Starting with A, B, P, Q, A', and B' as stated, consider

$$(AB, PQ) = \frac{|AQ||BP|}{|AP||BQ|} = \frac{1+z}{1-z}\frac{1-w}{1+w}$$

and

$$(A'B', PQ) = \frac{|A'Q||B'P|}{|A'P||B'Q|} = \frac{1+\phi(z)}{1-\phi(z)}\frac{1-\phi(w)}{1+\phi(w)}$$

where ϕ is our mapping from $\mathfrak{P}$ to $\mathfrak{K}$. Notice, however, that

$$1-\phi(z) = 1 - \frac{2z}{1+|z|^2} = \frac{1-2z+|z|^2}{1+|z|^2} = \frac{(1-z)^2}{1+z^2},$$

and similarly

$$1+\phi(z) = 1 + \frac{2z}{1+|z|^2} = \frac{1+2z+|z|^2}{1+|z|^2} = \frac{(1+z)^2}{1+z^2}.$$

This simplifies our situation so that

$$(A'B', PQ) = \frac{1+\phi(z)}{1-\phi(z)}\frac{1-\phi(w)}{1+\phi(w)} = \left(\frac{1+z}{1-z}\right)^2\left(\frac{1-w}{1+w}\right)^2 = (AB, PQ)^2.$$

It follows then that $\ln[(AB, PQ)] = \frac{1}{2}\ln[(A'B', PQ)]$. This leads to the following definition for distance in $\mathfrak{K}$.

Definition B.9 If A and B are points in $\mathfrak{K}$, the the (Klein) distance from A to B is

$$d_{\mathfrak{K}}(A, B) = \frac{1}{2}|\ln[(AB, PQ)]|,$$

where P and Q are the ideal points of $\overleftrightarrow{AB}$ (endpoints of the chord defining $\overleftrightarrow{AB}$).

Now that we have a definition for distance in $\mathfrak{K}$, we can give a few more definitions.

Definition B.10 Given a line l in $\mathfrak{K}$, *hyperbolic reflection* in $\mathfrak{K}$ is a mapping $R_{\mathfrak{K},l} : \mathfrak{K} \to \mathfrak{K}$ defined as follows: $R_{\mathfrak{K},l}(A) = A^*$ if $\overline{AA^*}$ is perpendicular to l and intersects l at M such that $d_{\mathfrak{K}}(AM) = d_{\mathfrak{K}}(A^*M)$.

Remark 13 *Notice that it follows, from our work on circle inversion in chapter 4.2, that euclidean circle inversions are hyperbolic reflections in* $\mathfrak{P}$ *and* $\mathbb{H}$.

Theorem B.11 *For any two lines* l *and* m *in* $\mathbb{H}$*, there exists an isometry of* $\mathbb{H}$ *which maps* l *to* m.

Proof: We have three main cases to consider here: first, that l and m each appear as a ray; second, that exactly one of l and m appears as a ray; and third, that l and m each appear as a semicircle. In each case we'll assume that l and m are distinct lines; if they are not distinct, then the identity is the desired isometry. Now, for our first case we can take

$$l = \{z \in \mathbb{H} : \Re(z) = a\} \quad \text{and} \quad m = \{z \in \mathbb{H} : \Re(z) = b\}$$

with $a \neq b$. Then the desired isometry is reflection across the line

$$n = \{z \in \mathbb{H} : \Re(z) = \frac{a+b}{2}\}$$

given by

$$R_n(z) = -\overline{\left(z - \frac{a+b}{2}\right)} + \frac{a+b}{2} = -\bar{z} + a + b.$$

See that for any $z \in l$,

$$R_n(z) = -a + \Im(z)i + a + b = b + \Im(z)i \in m.$$

For our second case, suppose the m appears as a ray,

$$m = \{z \in \mathbb{H} : \Re(z) = b\}$$

for some real b, and that l appears as a semicircle,

$$l = \{z \in \mathbb{H} : |z - w|^2 = r^2\}$$

for some w on the boundary L and real number r. Then by Theorem 4.33 there is a line n about which reflection maps l to m. Notice that we're using here the fact that if γ is a circle whose center lies on L, then inversion through γ, restricted to $\mathbb{H}$, is hyperbolic reflection through the line $\gamma \cap \mathbb{H}$.

Our third case follows from the second; supposing that l and m appear as semicircles, we can take a line n appearing as a semicircle. It follows from our second case that there is an isometry ρ_1 mapping l to n, and an isometry ρ_2 mapping m to n. Since the isometries of $\mathbb{H}$ form a group closed under composition, ρ_2^{-1} exists, and $\rho_2^{-1} \circ \rho_1$ is an isometry of $\mathbb{H}$ which will map l to n. ∎

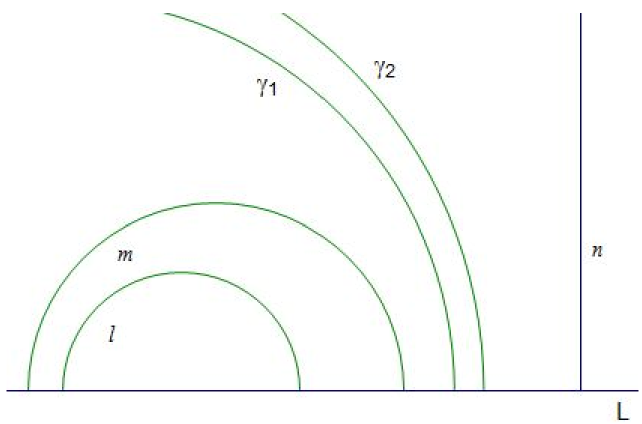


Figure B.2: Case 3 of Theorem B.11.

Regarding the third case considered in the proof of Theorem B.11, the isometries used, ρ_1 and ρ_2, are actually euclidean circle inversions restricted to $\mathbb{H}$, or hyperbolic reflections across γ_1 and γ_2 (see figure B.2). Since ρ_2 is then its own inverse, the isometry we gave, which maps l to n, could also be given as $\rho_2 \circ \rho_1$.

Definition B.12 Given a line l and a point P not on l, the *angle of parallelism* α of P with respect to l is the acute angle formed by line m, perpendicular to l and passing through P, with a line n which passes through P and is convergently parallel to l.

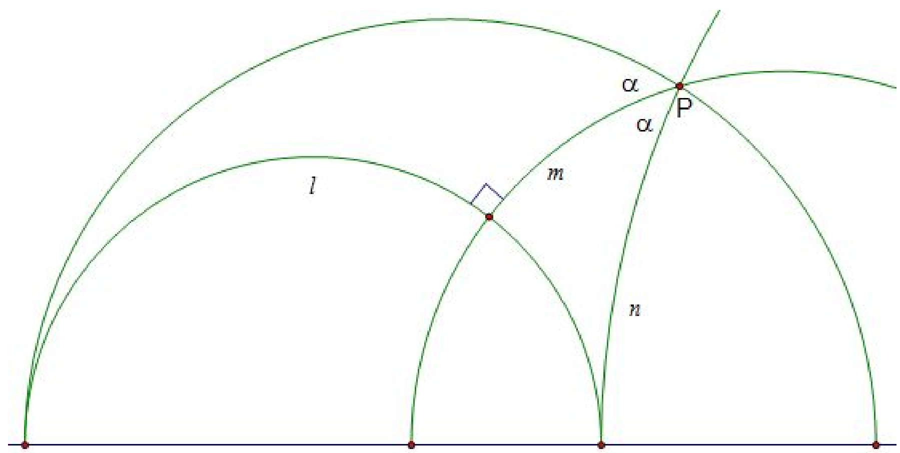


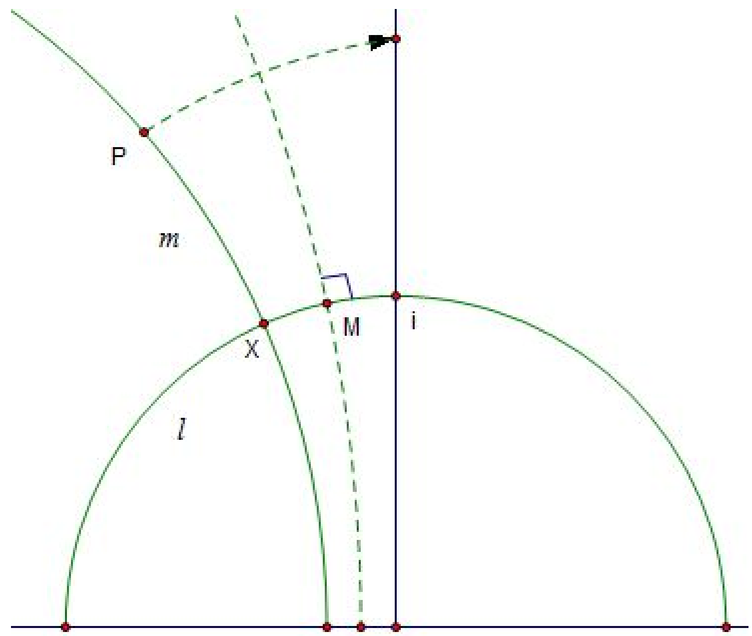
Figure B.3: The angle of parallelism α of P with respect to l in $\mathbb{H}$.

Theorem B.13 *Given a point P and a line l not containing P, the angle of parallelism at P with respect to l is determined only by the distance of P from l, such that $\sigma(d) = \cos^{-1}(\tanh(d))$ if the distance from P to l is d.*

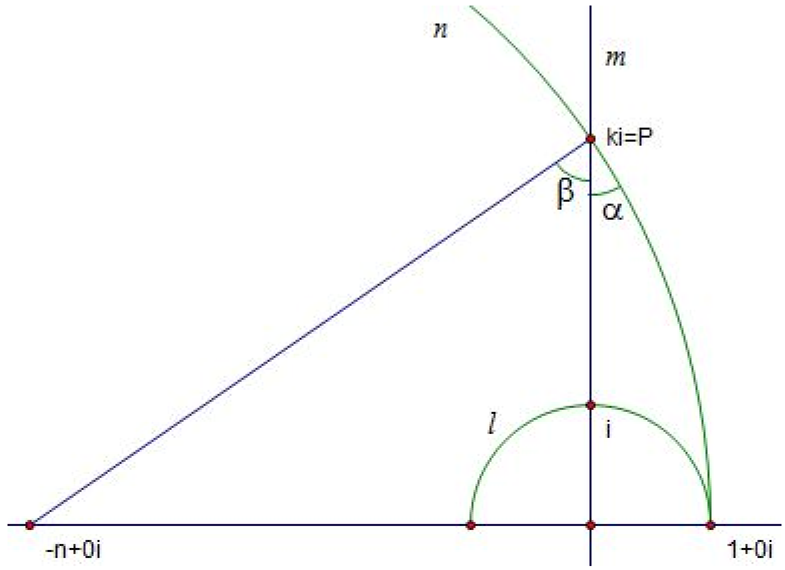
Proof: To prove this, we'll work in the half plane $\mathbb{H}$. Recall from Theorem B.11 that we can map any line l onto any other line by some isometry of $\mathbb{H}$; so wlog assume that l is given by

$$l = \{z = x + iy : x^2 + y^2 = 1, y > 0\}.$$

Letting m be the line passing through P perpendicular to l, m will intersect l at a point X. If P does not lie on the imaginary axis, then neither does X. If this is the case, then consider the point $Y = (0, i)$ on l; there exists a point M on $\overline{XY}$ which is the midpoint of this same segment. Reflection across the perpendicular bisector of $\overline{XY}$, the line perpendicular to l passing through M, will map the line m to the imaginary axis and map l to itself. Since hyperbolic reflection, though orientation reversing, preserves hyperbolic angles and distances, we can assume that P lies on the imaginary axis, so that $P = ki$ for some $k > 0$.



Now consider a line n passing through P which is convergently parallel to l as described in the figure below, where α is the angle of parallelism of P with respect to l.



Letting n be described by the euclidean circle with center $O = -n + 0i$ and radius r, we know that the (hyperbolic) distance from P to the line l is $\ln \frac{ki}{i} = \ln k$ since $P = ki$ lies on the imaginary axis, which is perpendicular to l. Denoting the distance by $d = \ln(k)$, we get that $k = e^d$. Also, we know that $n^2 + k^2 = r^2$, which gives us

$$\begin{aligned} n^2 + k^2 = r^2 &\Rightarrow (r-1)^2 + k^2 = r^2 \\ &\Rightarrow r = \frac{1+k^2}{2}, \end{aligned}$$

from which it follows that

$$\begin{aligned} \cos(\alpha) = \cos(90 - \beta) &= \sin(\beta) \\ &= \frac{n}{r} \\ &= \frac{r-1}{r} \\ &= \frac{\frac{1+k^2}{2} - 1}{\frac{1+k^2}{2}} \\ &= \frac{k^2-1}{k^2+1} \\ &= \frac{e^{2d}-1}{e^{2d}+1} = \tanh(d). \end{aligned}$$

Note that it can similarly be shown that $\sin(\alpha) = \frac{1}{\cosh(d)}$ and $\tan(\alpha) = \frac{1}{\sinh(d)}$. Then, defining a function σ by

$$\sigma(d) = \cos^{-1}(\tanh(d)) = \alpha,$$

we may find the angle of parallelism of a point with respect to a line with only the knowledge of the distance between the point and the line. ∎

Corollary B.1.2 *Take a hyperbolic line l and a point P not on l, and let m be the line perpendicular to l passing through p. If the ideal points of l are Q and R, and m intersects l at T, then $\angle QPT \cong \angle RPT$.*

Notice that the angles in this corollary are each the angle of parallelism of P with respect to l. Since the angle of parallelism of P with respect to l is determined by the distance from P to l, we have the desired result.

Definition B.14 Given distinct lines l and m in $\mathfrak{K}$ (or $\mathfrak{P}$, or $\mathfrak{H}$, or $\mathbb{H}$) which do not intersect, if l and m share an ideal point, then l and m are *convergently parallel*, and are called *divergently parallel* otherwise.

Theorem B.15 *If l is a line in $\mathfrak{K}$ appearing as a diameter, then hyperbolic reflection across l is the same as euclidean reflection (restricted to points in $\mathfrak{K}$) across the euclidean line m for which l is a segment.*

This follows from Theorem B.2.

Theorem B.16 *If l is a line in $\mathfrak{K}$, not appearing as a diameter, with pole P, then hyperbolic reflection across l is the restriction of the harmonic homology with center P and axis l to points in $\mathfrak{K}$.*

Proof: Let l be a line in $\mathfrak{K}$, not appearing as a diameter, with pole P, and let m be a line divergently parallel to l with ideal points Q and R. Also let Q' be the intersection of the euclidean line $\mathbf{e}\overleftrightarrow{PQ}$ with the boundary of the unit disk distinct from Q, and let R' be the intersection of $\mathbf{e}\overleftrightarrow{PR}$ with the boundary of the unit disk distinct from R. Then the line n_1 with ideal points Q and R' intersects l at a point M, and l intersects $\overleftrightarrow{QQ'}$ at a point S. See that $\angle QMS \cong \angle Q'MS$ since this is the angle of parallelism of M with respect to $\overleftrightarrow{QQ'}$. Similarly, m intersects $\overleftrightarrow{RR'}$ at a point T, and $\angle RMT \cong \angle R'MT$. Since $\angle RMT$ and $\angle Q'MT$ are vertical angles, we have

$$\angle R'MT \cong \angle RMT \cong \angle Q'MT \cong \angle QMT.$$

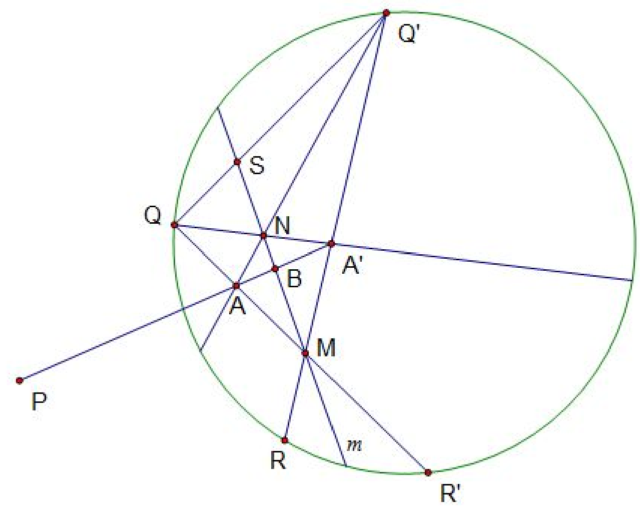
Now, we'll consider ϕ the perspectivity with center P mapping $\mathbf{e}\overleftrightarrow{QR'}$ to $\mathbf{e}\overleftrightarrow{Q'R}$. Clearly we have that ϕ maps n_1 to n_2, where n_2 is the line with ideal points Q' and R. Letting A be a point on n_1 distinct from M, let A' be the image of A under ϕ. Then by the definition of a perspectivity (Definition B.7) and Theorem B.2, $\overleftrightarrow{AA'} \perp m$. Letting the intersection of $\overleftrightarrow{AA'}$ with m be the point B, we have $\angle ABM \cong \angle A'BM$. Since we also have

$$\angle AMB \cong \angle QMT \cong \angle Q'MT \cong \angle A'MB$$

and $\overline{MB} \cong \overline{MB}$, we have by Theorem 2.21 that $\Delta AMB \cong \angle A'MB$, so $\overline{AB} \cong \overline{A'B}$. Since $\overleftrightarrow{AA'} \perp m$ and $\overline{AB} \cong \overline{A'B}$, we have that A' is the reflection of A across m.

Now, we still have to show that A and A' are harmonic conjugates with respect to the segment $\overline{PB}$; we need $(AA', PB) = 1$. First, see that we can talk about the perspectivity with center M; this maps P to itself, A to Q, B to S, and A' to Q'. Since perspectivities preserve the cross-ratio, we have

$$(AA', PB) = (QQ', PS).$$



Also, since A' is the reflection of A across m, and Q' is the reflection of Q across m, the line $\overleftrightarrow{A'Q}$ intersects $\overleftrightarrow{AQ'}$ at a point N on m. Then considering a perspectivity with center N, we have P mapped to P, A mapped to Q', M mapped to S, and A' mapped to Q, so we get

$$(AA', PB) = (Q'Q, PS).$$

However, we have the relation

$$(Q'Q, PS) = \frac{|Q'P||QS|}{|Q'S||QP|} = \frac{1}{(QQ', PS)},$$

so $(AA', PB)^2 = (QQ', PS)(Q'Q, PS) = 1$, which implies that (AA', PB) is its own reciprocal, hence $(AA', PB) = 1$. ∎

Construction of a midpoint of segment $\overline{AB}$ in $\mathfrak{K}$: Given a segment $\overline{AB}$, let l and m be lines perpendicular to $\overline{AB}$ at A and B respectively. The ideal points of l and m are P and Q such that $P - A - Q$, and R and S such that $R - B - S$. Construct chord $\overline{PS}$ which intersects $\overline{AB}$ at a point M. Since $\angle PMA \cong \angle SMB$, the angle of parallelism of M with respect to l is the same as the angle of parallelism of M with respect to m, so the distance from M to B is equal to the distance from M to A; hence, M is the midpoint of $\overline{AB}$.

The construction of the midpoint of a segment in our other models is similar; the following figures illustrate the construction of a midpoint in $\mathfrak{K}$, $\mathfrak{P}$, and $\mathbb{H}$.

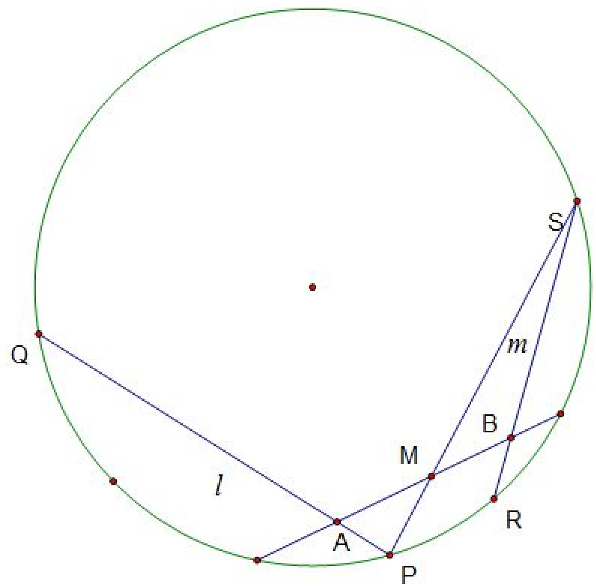


Figure B.4: Construction midpoints in the Klein disk model.

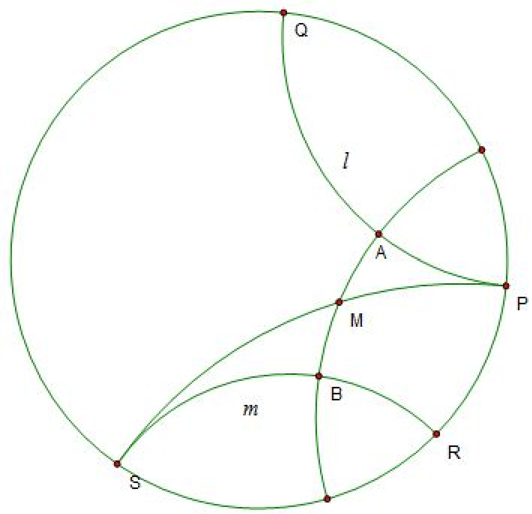


Figure B.5: Constructing midpoints in the Poincaré disk model.

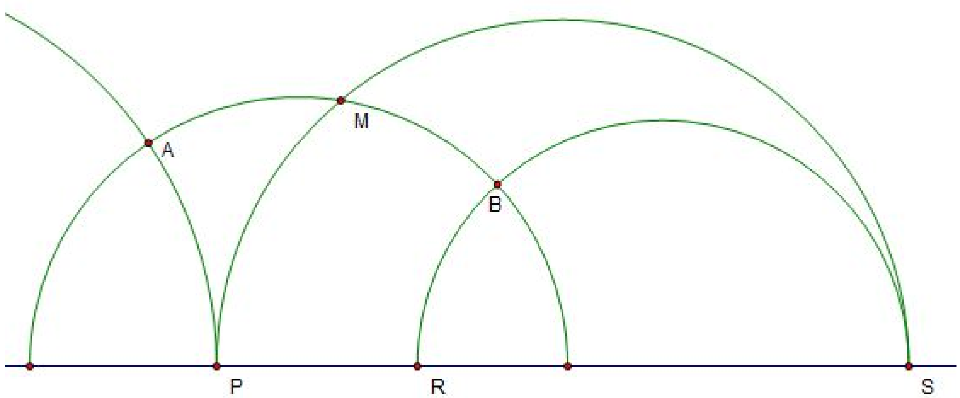


Figure B.6: Constructing midpoints in the half plane model.

B.2 Triangles

In our models of Hyperbolic Geometry, there are some special triangles that appear knows as *asymptotic triangles* (see figures B.7-B.9).

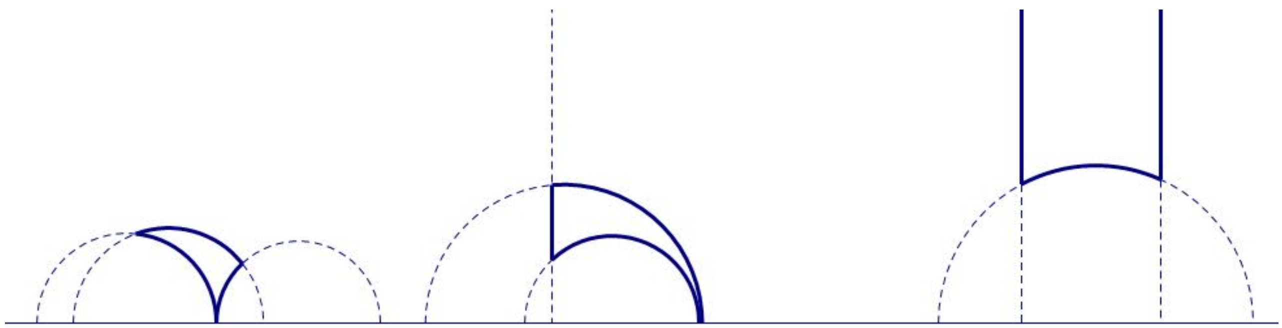


Figure B.7: Singly asymptotic triangles.

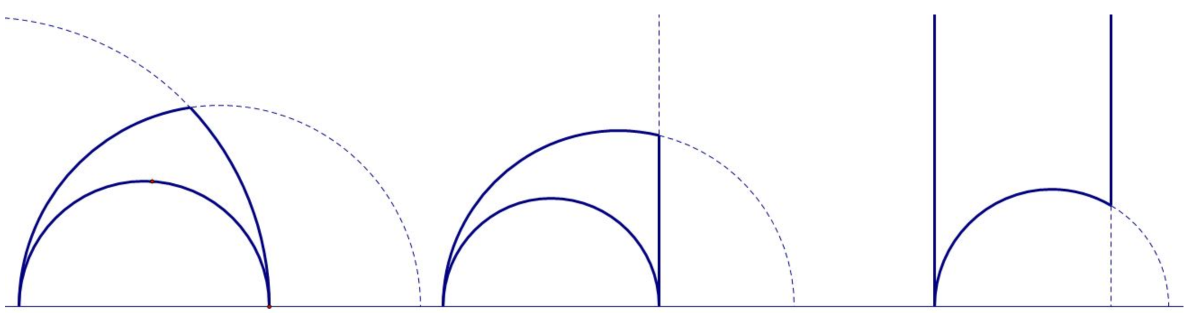


Figure B.8: Doubly asymptotic triangles.

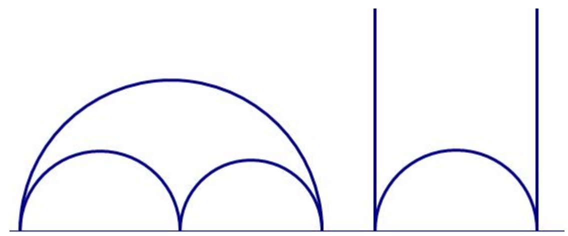


Figure B.9: Triply asymptotic triangles.

The triply asymptotic triangle, also called an *ideal triangle*, is given by three lines, each pair sharing a single ideal point; these ideal points serve as the vertices of such a triangle. In figure B.9 we see examples of this as illustrated in $\mathbb{H}$. We will be using these asymptotic triangles to define area of a region in the hyperbolic plane, which leads us to our next theorem, for which we'll need to measure angles in radians; define rad$\angle ABC$ to be the measure of $\angle ABC$ in radians.

Theorem B.17 *Every ideal triangle is congruent and has area* π*.*

Proof: We know that any line of $\mathbb{H}$ can be mapped to another by some isometry; it follows that any ideal triangle can also be mapped to another by some isometry, so all ideal triangles are congruent. Now, we can assume that the ideal triangle Δ under consideration is given by the lines

$l = \{z = x+yi : x = -1\}$, $m = \{z = x+yi : x = 1\}$, and $n = \{z = x+yi : |z|^2 = 1\}$,

so that the vertices of Δ are -1, 1, and ∞i. Then the triangular region whose area we claim to be π is

$$\mathring{\Delta} = \{z = x + yi : -1 < x < 1, y > \sqrt{1-x^2}\},$$

and the area of this region is given by

$$\int_{\mathring{\Delta}} dA = \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\infty} \frac{dydx}{y^2} = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \int_{0}^{\pi} \frac{\cos(\theta)d\theta}{\cos(\theta)} = \pi.$$

■

Using this, we can give a formula for the area of hyperbolic triangles in general.

Theorem B.18 *The area of a triangle ΔABC, where A, B, and C may be points of the hyperbolic plane or ideal points, is given by*

$$A(\Delta ABC) = \pi - \text{rad}\angle ABC - \text{rad}\angle BCA - \text{rad}\angle CAB.$$

Proof: By Theorem B.17 this holds for all ideal triangles, so we'll next consider a doubly asymptotic triangle. Again, by applying the appropriate isometry, we can assume that two of the lines used to define ΔABC are

$$l = \{z = x + yi : x = -1\} \text{ and } n = \{z = x + yi : |z|^2 = 1\},$$

and that one vertex of ΔABC is ∞i, so that the third line used to define our triangle is $m = \{z = x+yi : x = a\}$ for some $a \in (-1, 1)$. Supposing that $\text{rad}\angle BAC = \gamma \neq 0$, then $A = a + \sqrt{1-a^2}i = \cos(\gamma) + i\sin(\gamma)$. Then the area of ΔABC is given by

$$\int_{\mathring{\Delta} ABC} dA = \int_{-1}^{a}\int_{\sqrt{1-x^2}}^{\infty} \frac{dydx}{y^2} = \int_{-1}^{a} \frac{dx}{\sqrt{1-x^2}} = \int_{\gamma}^{\pi} \frac{\cos(\theta)d\theta}{\cos(\theta)} = \pi - \gamma.$$

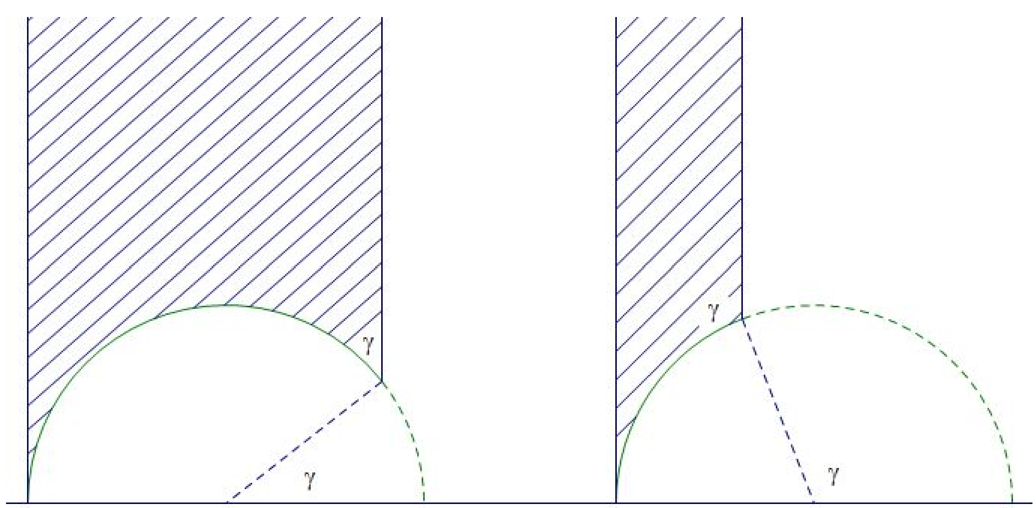


Figure B.10: Area of a doubly asymptotic triangle.

By a similar argument, if we have a singly asymptotic triangle ΔABC with non-zero interior angles rad$\angle ABC = \gamma$ and rad$\angle BCA = \beta$, then the area of this triangle is $\pi - \gamma - \beta$.

Finally, if ΔABC is not an asymptotic triangle, so that rad$\angle ABC = \beta$, rad$\angle BCA = \gamma$, and rad$\angle CAB = \alpha$ are all non-zero, then we can use singly (or doubly or triply) asymptotic triangles to find the area of ΔABC. Consider triangles ΔPAB and ΔPBC, where P is an ideal point of $\overleftrightarrow{AC}$, and let rad$\angle PBA = \theta$; note that rad$\angle PAB = \pi - \alpha$. Then

$$A(\Delta PAB) = \pi - (\pi - \alpha) - \theta) = \alpha - \theta$$

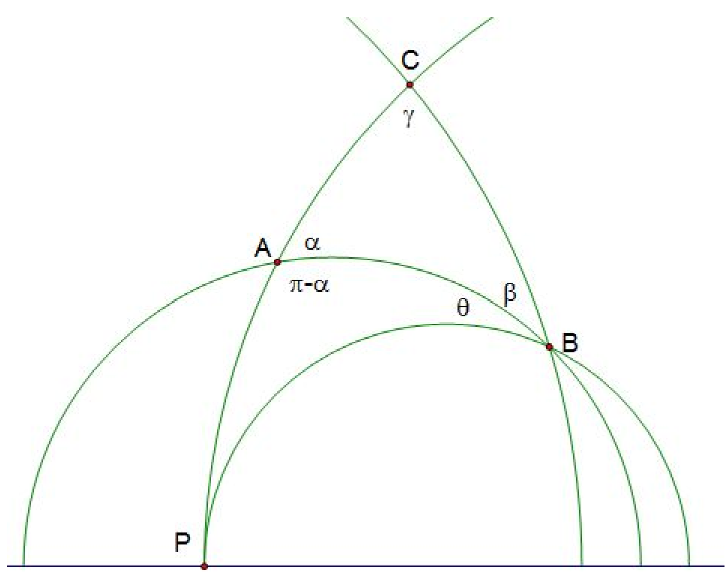
and

$$A(\Delta PBC) = \pi - \gamma - (\theta + \beta).$$

Subtracting one area from the other, we have that the area of ΔABC is

$$A(\Delta ABC) = A(\Delta PBC) - A(\Delta PAB) = (n-1)\pi - \gamma - (\theta + \beta) - (\alpha - \theta) = \pi - \gamma - \beta - \alpha.$$

∎



Remark 14 *Theorem B.18 is actually a special case of the Gauss-Bonnet theorem; applied to hyperbolic polygons in general, the theorem implies that the area of a hyperbolic n-gon P with vertices A_1, A_2, ..., A_n is given by*

$$A(P) = n\pi - \sum_{i=1}^{n} \mathrm{rad}\angle A_{i-1}A_iA_{i+1},$$

where $A_0 = A_n$ and $A_{n+1} = A_1$. This quantity is also called the defect of the polygon P; the defect of a hyperbolic n-gon is the euclidean angle sum of an n-gon less the angle sum of the hyperbolic n-gon.

B.3 Circles

Theorem B.19 *In the half plane model $\mathbb{H}$ every euclidean circle contained entirely in $\mathbb{H}$ is a hyperbolic circle, and every hyperbolic circle in $\mathbb{H}$ is a euclidean circle.*

Proof:

Consider a euclidean circle $\gamma = \{z = x + yi : |z - \omega|^2 = r^2\}$ for some $r > 0$ and ω such that $\Im(\omega) - r > 0$. Then consider the hyperbolic lines l and m given by

$$l = \{z = x + yi : |z - u|^2 = |\omega - u|^2 - r^2\}$$

and

$$m = \{z = x + yi : |z - v|^2 = |\omega - v|^2 - r^2\}$$

for some $u, v \in \mathbb{C}$ on the real axis; l and m are orthogonal to γ. Not only that, but l and m intersect at a point $A = z_0$. Given the definitions of l and m, we can solve

$$|z - u|^2 - |\omega - u|^2 = |z - v|^2 - |\omega - v|^2$$

for $\Re(z) = x$, finding that $\Re(z) = \Re(\omega)$. After some more algebra, one gets

$$\Im(z) = y = \sqrt{\Im(\omega)^2 - r^2}.$$

This point of intersection $z_0 = \Re(\omega) + \sqrt{\Im(\omega)^2 - r^2}i$ is the hyperbolic center of γ. To assure ourselves of this, let's return to the line l, and let n be a second line passing through z_0. Now, l appears as a semicircle, and so hyperbolic reflection across l is given by inversion about the circle describing l, and since l is orthogonal to γ, γ is fixed under hyperbolic reflection across l. Not only that, but the segment $\overline{AB}$ of m interior to γ, with A and B on γ, must also be sent to some segment $\overline{A'B'}$ interior to γ with A' and B' on γ. Since hyperbolic reflection preserves hyperbolic distance, $d_{\mathbb{H}}(A, B) = d_{\mathbb{H}}(A'B')$. Since n was chosen arbitrarily as a line passing through z_0, this is true of all lines passing through z_0, so γ is a hyperbolic circle with center $z_0 = \Re(w) + (\Im(w)^2 - r^2)^{\frac{1}{2}}i$.

For the second direction, consider a hyperbolic circle δ with center w and radius s, and a euclidean circle γ with center v and radius r. There exists a line l such that hyperbolic reflection across l maps the point $z_0 = \Re(v) + (\Im(v)^2 - r^2)^{\frac{1}{2}}$ to w; let γ' and v' be the images of γ and v under this reflection. Now, euclidean dilation preserves euclidean circles, and there is some $k > 0$ such that dilation with center w and ratio k maps γ' to γ'', where γ'' intersects δ at at least one point, z. Now, from the previous discussion γ'' is also a hyperbolic circle with center w. Since γ'' and δ both have center w and have at least one point in common, they must be the same circle; hence, δ is also a euclidean circle.

∎

It follows from the previous theorem that euclidean circles in $\mathfrak{P}$ are hyperbolic circles, and visa versa. In fact, another approach to proving Theorem B.19 begins in the Poincaré disk by taking a euclidean circle in $\mathfrak{P}$ whose center is the origin. While distances in $\mathfrak{P}$ are distorted as we move away from the origin, by starting with a circle centered at the origin we have that the hyperbolic distance from any point on this circle to the center is "evenly" distorted; while the hyperbolic radius (since this is in fact a hyperbolic circle as well as a euclidean circle) and euclidean radius of this circle are not the same, the hyperbolic and euclidean centers are. In the figure below we see several hyperbolic circles, all of which have the same (hyperbolic) radius.

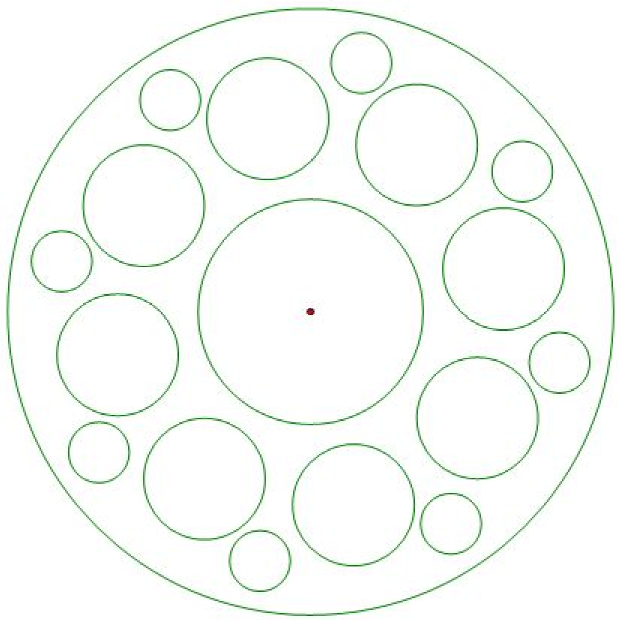


Figure B.11: Circles in the Poincaré disk.

Each of the circles in Figure (B.11) is actually a reflection of the inner-most circle, that centered at the origin. Since hyperbolic reflection (euclidean circle inversion restricted to the interior of the unit disk) maps circles to circles and preserves hyperbolic distance, each euclidean circle in figure (B.11) is again a hyperbolic circle with a different center, though all have equal radii.

Bringing this into the half plane model, $\mathbb{H}$, involves certain stereographic projections, the first mapping $\mathfrak{P}$ onto the northern hemisphere of the unit sphere from the south-pole (or onto the southern hemisphere from the north-pole); this is a model of Hyperbolic Geometry discuss earlier. Moving from the hemisphere to the half plane, we stereographically project the hemisphere, from a point on the equator, to the half-plane; this mapping is described in the figures below.

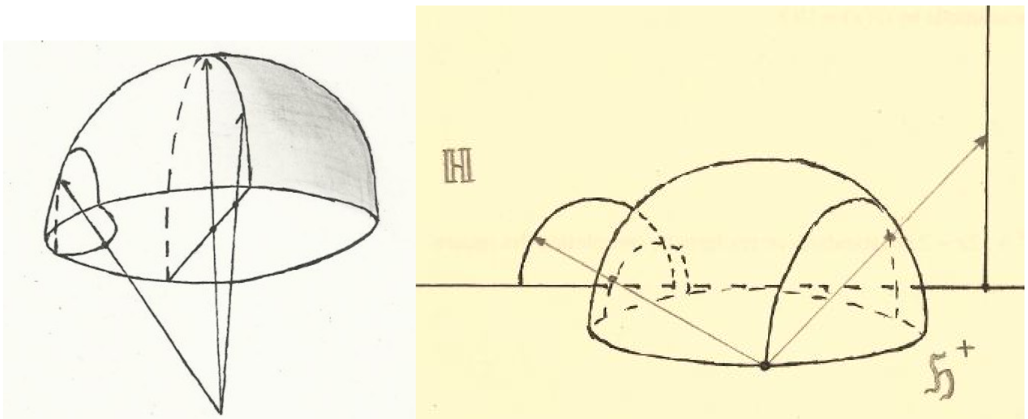


Figure B.12: Projections

In the figure above, we see the projection described; on the left, mapping the Poincaré disk to the northern hemisphere of the unit sphere $\mathfrak{H}^+$, and then on the right projecting $\mathfrak{H}^+$ to the half plane model $\mathbb{H}$.

Construction of a Circle in $\mathfrak{P}$ given a center and radius: Suppose we're given points A and B, and we want to find the hyperbolic circle γ with center A and radius $\overline{AB}$. Let l be the line perpendicular to $\overline{AB}$ at A, and let C be the hyperbolic reflection of B across l; $\overline{BC}$ should be a diameter of the circle to be constructed. Now, since l passes through A, the circle we want to construct should be fixed under reflection across l (l passes through the center of the desired circle). Since $\overline{BC}$ is a diameter of γ, γ will meet $\overleftrightarrow{AB}$ at right angles, so the euclidean circles representing γ and $\overleftrightarrow{AB}$ should be orthogonal; we may then construct euclidean tangents to $\overleftrightarrow{BC}$ at B and C, which meet at some point D. This point D is the euclidean center of γ, so constructing the euclidean circle with center D and radius $\overline{DB}$ gives us γ, the hyperbolic circle with center A and radius $\overline{AB}$.

Appendix C

Hyperbolic Functions

C.1 Hyperbolic Functions: Definitions and Properties

The hyperbolic functions are defined as follows:

$$
\begin{aligned}
\textit{hyperbolic cosine} &: \cosh(x) = \frac{e^x + e^{-x}}{2} = \frac{e^{2x}+1}{2e^x} \\
\textit{hyperbolic sine} &: \sinh(x) = \frac{e^x - e^{-x}}{2} = \frac{e^{2x}-1}{2e^x} \\
\textit{hyperbolic tangent} &: \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1} \\
\textit{hyperbolic cotangent} &: \coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1} \\
\textit{hyperbolic secant} &: \mathrm{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}} = \frac{2e^x}{e^{2x}+1} \\
\textit{hyperbolic cosecant} &: \mathrm{scsh}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}} = \frac{2e^x}{e^{2x}-1}
\end{aligned}
$$

Some properties of the hyperbolic functions and their inverses include:

1. $\cosh(0) = 1$

2. $\sinh(0) = 0$

3. $\sinh(-x) = -\sinh(x)$

4. $\cosh(-x) = \cosh(x)$

5. $\sinh(2x) = 2\cosh(x)\sinh(x)$

6. $\cosh^2(x) - \sinh^2(x) = 1$

$$\begin{aligned}
\cosh^2(x) - \sinh^2(x) &= \frac{e^4x + 2e^{2x} + 1}{4e^{2x}} - \frac{e^4x - 2e^{2x} + 1}{4e^{2x}} \\
&= \frac{4e^{2x}}{4e^{2x}} \\
&= 1
\end{aligned}$$

7. $\operatorname{sech}^2 x + \tanh^2(x) = 1$

$$\begin{aligned}
\operatorname{sech}^2 x + \tanh^2(x) &= \frac{1 + \sinh^2(x)}{\cosh^2(x)} \\
&= \frac{\cosh^2(x)}{\cosh^2(x)} \\
&= 1
\end{aligned}$$

8. $\coth^2(x) - \operatorname{scsh}^2(x) = 1$

$$\begin{aligned}
\coth^2(x) - \operatorname{scsh}^2(x) &= \frac{\cosh^2(x) - 1}{\sinh(x)} \\
&= \frac{\sinh^2(x)}{\sinh^2(x)} \\
&= 1
\end{aligned}$$

9. $\cosh(x+y) = \sinh(x)\sinh(y) + \cosh(x)\cosh(y)$

$$\begin{aligned}
\cosh(x+y) &= \frac{e^{2x}e^{2y}+1}{2e^xe^y} \\
&= \frac{2e^{2x}e^{2y}+2}{4e^xe^y} \\
&= \frac{2e^{2x}e^{2y}-e^{2x}-e^{2y}+e^{2x}+e^{2y}+2}{4e^xe^y} \\
&= \frac{e^{2x}e^{2y}-e^{2x}-e^{2y}+1}{4e^xe^y} + \frac{e^{2x}e^{2y}+e^{2x}+e^{2y}+1}{4e^xe^y} \\
&= \frac{e^{2x}-1}{2e^x}\frac{e^{2y}-1}{2e^y} + \frac{e^{2x}+1}{2e^x}\frac{e^{2y}+1}{2e^y} \\
&= \sinh(x)\sinh(y) + \cosh(x)\cosh(y)
\end{aligned}$$

10. $\cosh(2x) = \sinh^2(x) + \cosh^2(x)$

11. $\sinh(x+y) = \cosh(x)\sinh(y) + \sinh(x)\cosh(y)$

12. $\sinh(2x) = 2\cosh(x)\sinh(x)$

13. $\cosh^{-1}(x) = \text{arccosh}(x) = \ln(x + \sqrt{x^2-1})$; $x \geq 1$

Take $y = \cosh(x) = \frac{e^x+e^{-x}}{2}$. Then, solving for x,

$$\begin{aligned}
2y = e^x + e^{-x} &\Rightarrow 0 = e^x - 2y + e^{-x} \\
&\Rightarrow 0 = e^{2x} - 2ye^x + 1 \\
&\Rightarrow e^x = \frac{2y + \sqrt{4y^2-4}}{2} = y + \sqrt{y^2-1} \\
&\Rightarrow x = \ln(y + \sqrt{y^2-1})
\end{aligned}$$

14. $\sinh^{-1}(x) = \text{arcsinh}(x) = \ln(x + \sqrt{x^2+1})$

15. $\tanh^{-1}(x) = \text{arctanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$; $-1 < x < 1$

Take $y = \tanh(x)$, and solving for x,

$$\begin{aligned} y(e^{2x}+1) = e^{2x}-1 \quad &\Rightarrow \quad 0 = e^{2x}(1-y)-(1+y) \\ &\Rightarrow \quad e^{2x} = \frac{1+y}{1-y} \\ &\Rightarrow \quad x = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right) \end{aligned}$$

C.2 Computations

In our discussion of Fuchsian groups, some calculations relating to the distance function in $\mathbb{H}$ were said to follow from

$$d_{\mathbb{H}}(z,w) = \ln\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|};$$

this is shown here.

1. Note that $\Re(z-\bar{w}) = \Re(z-w)$, so

$$|z-\bar{w}|^2-|z-w|^2 = \Re(z-\bar{w})^2+\Im(z-\bar{w})^2-\Re(z-w)^2-\Im(z-w)^2 = \Im(z-\bar{w})^2-\Im(z-w)^2$$

$$\begin{aligned} \cosh d_{\mathbb{H}}(z,w) &= \frac{e^{d_{\mathbb{H}}(z,w)}+e^{-d_{\mathbb{H}}(z,w)}}{2} \\ &= \frac{1}{2}\left(\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|}+\frac{|z-\bar{w}|-|z-w|}{|z-\bar{w}|+|z-w|}\right) \\ &= \frac{1}{2}\left(\frac{2|z-\bar{w}|^2+2|z-w|^2}{|z-\bar{w}|^2-|z-w|^2}\right) \\ &= 1+\frac{2|z-w|^2}{|z-\bar{w}|^2-|z-w|^2} \\ &= 1+\frac{2|z-w|^2}{\Im(z-\bar{w})^2-\Im(z-w)^2} \\ &= 1+\frac{|z-w|^2}{2\Im(z)\Im(w)} \end{aligned}$$

2.

$$\begin{aligned}
\sinh\left(\frac{1}{2}d_{\mathbb{H}}(z,w)\right) &= \frac{1}{2}\frac{e^{d_{\mathbb{H}}(z,w)}-1}{e^{d_{\mathbb{H}}(z,w)/2}} \\
&= \frac{1}{2}\frac{\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|}-1}{\sqrt{\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|}}} \\
&= \frac{|z-w|}{\sqrt{|z-\bar{w}|^2-|z-w|^2}} \\
&= \frac{|z-w|}{2\sqrt{\Im(z)\Im(w)}}
\end{aligned}$$

3.

$$\begin{aligned}
\cosh\left(\frac{1}{2}d_{\mathbb{H}}(z,w)\right) &= \frac{1}{2}\frac{e^{d_{\mathbb{H}}(z,w)}+1}{e^{d_{\mathbb{H}}(z,w)/2}} \\
&= \frac{1}{2}\frac{\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|}+1}{\sqrt{\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|}}} \\
&= \frac{|z-\bar{w}|}{\sqrt{|z-\bar{w}|^2-|z-w|^2}} \\
&= \frac{|z-\bar{w}|}{2\sqrt{\Im(z)\Im(w)}}
\end{aligned}$$

4. From the last two, we have

$$\begin{aligned}
\tanh\left(\frac{1}{2}d_{\mathbb{H}}(z,w)\right) &= \frac{\sinh\left(\frac{1}{2}d_{\mathbb{H}}(z,w)\right)}{\cosh\left(\frac{1}{2}d_{\mathbb{H}}(z,w)\right)} \\
&= \frac{|z-w|}{|z-\bar{w}|}
\end{aligned}$$

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